Stable bundles on non-Kähler manifolds with transversally Kähler foliations

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Holomophic foliations and complex dynamics

14 June 2012, Poncelet Laboratory, Moscow, Russia

Motivation.

QUESTION: How to study non-Kähler complex manifolds?

THEOREM: (Harvey-Lawson) Let M be a compact complex n-manifold not admitting a Kähler structure. **Then there exists a positive $(n-1,n-1)$-current T which is an $(n-1,n-1)$-part of an exact current.**

THEOREM: (Harvey-Lawson) Let M be a compact complex non-Kähler surface. **Then M admits a positive, exact (1,1)-current.**

DEFINITION: Kähler rank of a complex manifold M is maximal rank of a positive, closed (1,1)-form on M.

Marco Brunella: classification of non-Kähler surfaces according to their Kähler rank.

We are interested in n-manifolds admitting positive, exact forms of rank $n-1$, which are transversally Kähler with respect to a 1-dimensional foliation.

Plan.

1. Motivation: Manifolds with transversally Kähler foliations (examples).

2. Immediate application: Classification of complex subvarieties.

3. Geometry of Vaisman manifolds. Structure theorem and transversal foliations. Application to complex subvarieties.

4. Stable bundles on non-Kähler manifolds. Gauduchon metrics and stability.

5. Coherent sheaves on Vaisman manifolds. Filtered sheaves and applications of Yang-Mills theory.

Transversally Kähler forms.

DEFINITION: A model situation. Let M be a compact complex manifold, $\Sigma \subset TM$ a holomorphic 1-dimensional foliation, generated by a nowhere vanishing holomorphic vector field s, and ω_0 a closed semipositive (1,1)-form on M. We say that ω_0 is **transversally Kähler**, if Σ is the null-space of ω_0, **equivariant**, if $\operatorname{Lie}_s \omega_0 = 0$, and **transversally Kähler exact** if ω_0 is exact: $\omega_0 = d\theta$.

REMARK: A manifold admitting a transversally Kähler exact form **is never Kähler**. Indeed, if ω is a Kähler form, we would have $\int_M \omega_0 \wedge \omega^{\dim M - 1} > 0$, which is impossible by Stokes' theorem, because $\omega_0 \wedge \omega^{\dim M-1} = d(\theta \wedge \omega^{\dim M-1})$.

EXAMPLE: A classical Hopf surface is $H := \mathbb{C}^2 \backslash 0 / \mathbb{Z}$, where $\mathbb{Z}$ acts as a multiplication by a complex number λ, $|\lambda| > 1$.

OBSERVATION: H is diffeomorphic to $S^1 \times S^3$, and fibered over $\mathbb{C}P^1$ with fiber $\mathbb{C}^* / \langle \lambda \rangle$.

CLAIM: Let $\pi: H \longrightarrow \mathbb{C}P^1$ be the standard projection, and $\omega_0 := \pi^* \omega_{\mathbb{C}P^1}$ be a pullback of the Fubini-Study form. Clearly, **ω_0 is exact, because $H^2(H) = 0$** (by Künneth formula). Therefore, **H admits a transversally Kähler, exact form.**

Locally trivial elliptic fibrations.

DEFINITION: A principal elliptic fibration M is a complex manifold equipped with a free holomorphic action of a 1-dimensional compact complex torus T.

Such a manifold is fibered over M/T, with fiber T.

REMARK: It is a principal T-bundle: all fibers are identified with T, with T acting on fibers freely.

DEFINITION: Let $M \stackrel{\pi}{\longrightarrow} X$ be a principal elliptic fibration, M compact. We say that M is **positive elliptic fibration**, if for some Kähler class ω on X, $\pi^*\omega$ is exact. **("Kähler class" is a cohomology class of a Kähler form).**

EXAMPLE: The classical Hopf surface introduced earlier.

EXAMPLE: A more general example is given by Tot$(L^*)/\langle\mathbb{Z}\rangle$, where L is an ample line bundle. Such manifold is called **a regular Vaisman manifold**. It is positive, because $\pi^*(c_1(L)) = 0$, and $c_1(L)$ is a Kähler class.

Calabi-Eckmann manifolds

Calabi-Eckmann manifolds.
Fix $\alpha \in \mathbb{C}$, α non-real, $|\alpha| > 1$. Consider a subgroup

$$G := \{e^t \times e^{\alpha t} \subset \mathbb{C}^* \times \mathbb{C}^*, \quad t \in \mathbb{C}\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

within $\mathbb{C}^* \times \mathbb{C}^*$. It is clearly co-compact and closed, with $\mathbb{C}^* \times \mathbb{C}^*/G$ being an elliptic curve $\mathbb{C}^*/\langle\alpha\rangle$.

Now, let $M := (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/G$, with $G \subset \mathbb{C}^* \times \mathbb{C}^*$ acting on $(\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)$ by $(t_1, t_2)(x, y) \longrightarrow (t_1 x, t_2 y)$. Clearly, M is fibered over

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} = (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/\mathbb{C}^* \times \mathbb{C}^*$$

with a fiber $\mathbb{C}^* \times \mathbb{C}^*/G$, which is an elliptic curve. Its total space M **the Calabi-Eckmann manifold**. It is diffeomorphic to $S^{2n-1} \times S^{2m-1}$.

REMARK: The map $M \longrightarrow \mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$ **is a principal elliptic fibration**.

REMARK: The pullback of a Kähler form from $\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$ to M **is exact**, because $H^2(M) = 0$ (by Künneth formula).

Irregular and quasi-regular foliations

DEFINITION: A foliation is called **quasi-regular** if all its leaves are compact. If this is not so, it is called **irregular**. A foliation is called **regular** if all its leaves are compact, and the leaf space is smooth.

So far, in our examples all foliations were quasi-regular. Let's have some irregular examples.

REMARK: Calabi-Eckmann manifolds were generalized by Lopez de Medrano, Verjovsky and Meersseman. The complex structure on Calabi-Eckmann can be deformed together with the foliation, giving **an equivariant transversally Kähler manifold with a foliation having non-compact leaves** ("LVM-manifolds").

REMARK: If (M, Σ, ω_0) is a manifold equipped with a transversally Kähler, exact form, and $Z \subset M$ is a complex subvariety, **then Z is a union of closures of leaves of Σ.**

For more examples, we define **Vaisman manifolds**.

LCK manifolds

DEFINITION: Let M be a complex manifold, $\tilde{M} \longrightarrow M$ its covering, $\tilde{M}/\Gamma = M$, and Γ the **monodromy group** freely acting on $\tilde{M}$. Assume that $\tilde{M}$ is Kähler, and Γ acts on $\tilde{M}$ by homotheties. Then M is called **locally conformally Kähler** (LCK).

EXAMPLE: A classical Hopf manifold $\mathbb{C}^n \backslash 0/\langle A\rangle$, $A(x) = qx$, $|q| > 1$, is obviously LCK.

DEFINITION: Let M be an LCK manifold, $(\tilde{M}, \tilde{\omega})$ its covering. An **LCK metric** on M is a metric conformal to $\tilde{\omega}$.

REMARK: If $\omega = f\tilde{\omega}$, one has $d\omega = df \wedge \omega'$. Therefore, **an LCK metric satisfies $d\omega = \theta \wedge \omega$,** for some closed 1-form θ.

REMARK: If $\theta = df$, the form $e^{-f}\omega$ is Kähler.

Hermitian metrics on LCK manifolds

CLAIM: If (M, I, ω) is a Hermitian manifold with $d\omega = \theta\wedge\omega$, for some closed 1-form θ, **then (M, I) is LCK.**

Proof: For some covering of M, the pullback of θ is exact, and then **the pullback of ω is conformal to a Kähler form: $\theta = df$, then $d(e^{-f}\omega) = e^{-f}\omega\wedge\theta - e^{-f}\omega\wedge\theta = 0$.** ■

DEFINITION: A Hermitian manifold (M, I, ω) is called **LCK** if $d\omega = \theta\wedge\omega$, for some closed 1-form θ.

DEFINITION: The form θ is called **the Lee form** of an LCK-manifold. The dual vector field $\theta^\sharp$ – **the Lee field**.

Vaisman manifolds

DEFINITION: An LCK manifold (M, ω, θ) is called **Vaisman** ("generalized Hopf") if $\nabla_{LC}\theta = 0$, where ∇_{LC} is the Levi-Civita connection.

THEOREM: (Kamishima-Ornea) Let M be an LCK manifold equipped with conformal a holomorphic flow. Assume that this flow acts by non-isometric homotheties on the Kähler covering. **Then M is Vaisman.**

EXAMPLE: The classical Hopf manifolds are obviously Vaisman.

THEOREM: Let (M, ω, θ) be a Vaisman manifold. Then **the form $\omega_0 := \omega - \theta \wedge I(\theta)$ is semi-positive and exact: $\omega_0 = d(I\theta)$.** Moreover, **the foliation Σ generated by $\theta^\sharp$ is holomorphic,** and ω_0 is equivariant and transversally Kähler with respect to Σ.

REMARK: **Vaisman manifolds are build from Sasakian manifolds;** there is an inexhaustible supply of those. More of this later in this talk.

Oeljeklaus-Toma manifolds

Let K be a number field which has $2t$ complex embedding denoted $\tau_i, \overline{\tau}_i$ and s real ones denoted σ_i, $s > 0$, $t > 0$.

Let $\mathcal{O}_K^{*,+} := \mathcal{O}_K^* \cap \bigcap_i \sigma_i^{-1}(\mathbb{R}^{>0})$. Choose in $\mathcal{O}_K^{*,+}$ a free abelian subgroup $\mathcal{O}_K^{*,U}$ of rank s such that the quotient $\mathbb{R}^s/\mathcal{O}_K^{*,U}$ is compact, where $\mathcal{O}_K^{*,U}$ is mapped to $\mathbb{R}^t$ as $\xi \longrightarrow \Big(\log(\sigma_1(\xi)), ..., \log(\sigma_t(\xi))\Big)$. Let $\Gamma := \mathcal{O}_K^{+} \rtimes \mathcal{O}_K^{*,U}$.

DEFINITION: An Oeljeklaus-Toma manifold is a quotient $\mathbb{C}^t \times H^s/\Gamma$, where $\mathcal{O}_K^{+}$ acts on $\mathbb{C}^t \times H^t$ as

$$\zeta(x_1, ..., x_t, y_1, ..., y_s) = (x_1 + \tau_1(\zeta), ..., x_t + \tau_t(\zeta), y_1 + \sigma_1(\zeta), ..., y_s + \sigma_s(\zeta)),$$

and $\mathcal{O}_K^{*,U}$ as $\xi(x_1, ..., x_t, y_1, ..., y_s) = (x_1, ..., x_t, \sigma_1(\xi)y_1, ..., \sigma_t(\xi)y_t)$

THEOREM: (Oeljeklaus-Toma) The OT-manifold $M := \mathbb{C}^t \times H^s/\Gamma$ **is a compact complex manifold,** without any non-constant meromorphic functions. When $t = 1$, it is locally conformally Kähler. When $s = 1, t = 1$, it is an Inoue surface of class S^0.

THEOREM: (Ornea-V.) Let M be an OT-manifold, $t = 1$. **Then M is equipped with a holomorphic 1-dimensional foliation and an equivariant, transversally Kähler, exact form.**

Complex subvarieties in transversally Kähler manifolds

THEOREM: Let (M, Σ, ω_0) be a manifold, equipped with a transversally Kähler, exact form, and $Z \subset M$ a complex subvariety. **Then Z is a union of leaves of Σ.**

Proof: Suppose that Z is transversal to Σ at a point z. Then all eigenvalues of $\omega_0|_Z$ at z are positive, hence $\int_Z \omega_0^k > 0$, where $k := \dim Z$. This is impossible by Stokes' theorem, because ω_0^k is exact. ■

COROLLARY: Let $M \xrightarrow{\pi} X$ be a positive, principal elliptic fibration (such as Calabi-Eckmann, or Hopf manifold). **Then all positive-dimensional subvarieties of M are of form $\pi^{-1}(Z)$,** for some complex subvariety $Z \subset M$.
■

THEOREM: (Ornea-V.) **An OT-manifold for $t = 1$ has no non-trivial complex subvarieties.**

The proof of this theorem requires a bit of number theory.

Sasakian manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and (ω, I) a Kähler structure on $M \times \mathbb{R}^{>0}$. Suppose that ω is **homogeneous**: $\Psi_q^*\omega = q^2 g$, where $\Psi_q(m,t) = (m, qt)$, and I is Ψ_q-invariant. Then M is called **Sasakian**, and $M \times \mathbb{R}^{>0}$ its **Kähler cone**.

Sasakian geometry is an odd-dimensional counterpart to Kähler geometry

Example: An odd-dimensional sphere S^{2n-1} is Sasakian. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{C}^n \backslash 0$ has the standard Kähler form $\sqrt{-1}\sum_{i=1}^n dz_i \wedge d\overline{z}_i$ which is obviously homogeneous.

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.

Quasiregular Sasakian manifolds.

Definition: Given a contact manifold (M, θ) with a Riemannian structure g, the dual vector field $\theta^\sharp$ is called **the Reeb field** of (M, θ, g).

Remark: For any Sasakian manifold, **the Reeb field generates a flow of diffeomorphisms acting on M by contact isometries**. This is obvious from the definition, because the Reeb field $\theta^\sharp = It\frac{d}{dt}$ acts by holomorphic isometries on the Kähler cone.

Definition: A Sasakian manifold M is called **quasiregular** if all orbits of the Reeb flow are compact. The space of orbits of the Reeb flow is a complex orbifold. **Every quasiregular Sasakian manifold is a total space of S^1-bundle over a complex orbifold.**

This is easy to see, because the quotient of M over the Reeb flow is the same as the quotient of CM over its complexification, generated by $\theta^\sharp$ and $I\theta^\sharp$.

Examples of Sasakian manifolds.

Example: Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $CX \subset \mathbb{C}^{n+1}\backslash 0$ the corresponding cone. The cone CX is obviously Kähler and homogeneous, hence **the intersection $CX \cap S^{2n-1}$ is Sasakian.** This intersection is an S^1-bundle over X. This construction gives many interesting contact manifolds, including Milnor's exotic 7-spheres, which happen to be Sasakian.

REMARK: In other words, **a link of a homogeneous singularity is always Sasakian.**

REMARK: **Every quasiregular Sasakian manifold is obtained this way,** for some Kähler metric on $\mathbb{C}^{n+1}$ (Ornea-V., arXiv:math/0609617).

REMARK: All 3-dimensional Sasakian manifolds are quasiregular (H. Geiges, 1997, F. Belgun, 2000).

REMARK: **Every Sasakian manifold is diffeomorphic to a quasiregular one** (Ornea-V., arXiv:math/0306077)

Structure theorem for Vaisman manifolds

THEOREM: (Ornea-V.) **Every Vaisman manifold is obtained as $C(X)/\mathbb{Z}$,** where X is Sasakian, $\mathbb{Z} = \Big\langle (x,t) \mapsto (\varphi(x), qt) \Big\rangle$, $q > 1$, and φ is a Sasakian automorphism of X. Moreover, the triple (X, φ, q) is unique.

REMARK: This is equivalent to the **existence of a Riemannian submersion $M \longrightarrow S^1$,** with Sasakian fibers.

REMARK: **This construction gives an equivalence** between the category of Vaisman manifolds, and the category of triples (X, φ, q) (Sasakian manifold, a Sasakian automorphism, number).

Gauduchon metrics

DEFINITION: A Hermitian metric ω on a complex n manifold is called **Gauduchon** if $\partial\overline{\partial}\omega^{n-1} = 0$.

THEOREM: (P. Gauduchon, 1978) Let M be a compact, complex manifold, and h a Hermitian form. **Then there exists a Gauduchon metric conformally equivalent** to h, and it is unique, up to a constant multiplier.

REMARK: If ω is Gauduchon, then (by Stokes' theorem) $\int_M \omega^{n-1}\partial\overline{\partial} f = 0$ for any f. The curvature Θ_L of a holomorphic line bundle L is well-defined up to $\partial\overline{\partial}\log|h|$, where h is a conformal factor. Therefore, **for any line bundle L, the quantity $\deg_\omega L := \int_M \omega^{n-1} \wedge \Theta_L$ is well defined.**

REMARK: Unlike the Kähler case, $\deg_\omega L$ is a holomorphic invariant of L, and **not topological.**

DEFINITION: Given a torsion-free coferent sheaf F of rank r, let $\det F := \Lambda^r F^{**}$. From algebraic geometry it is known that $\det F$ is a line bundle. Define **the degree** $\deg_\omega F := \deg_\omega \det F$.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional Gauduchon manifold (M,ω). Let

$$\operatorname{slope}(F) := \frac{\deg_\omega F}{\operatorname{rank}(F)}$$

A torsion-free sheaf F is called **stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if $\Theta_B \wedge \omega^{n-1} = \operatorname{slope}(F) \cdot \operatorname{Id}_B \cdot \omega^n$, where Θ_B is its curvature.

THEOREM: (**Kobayashi-Hitchin correspondence**; Donaldson, Buchsdahl, Uhlenbeck-Yau, Li-Yau, Lübke-Teleman): Let B be a holomorphic vector bundle. **Then B admits Yang-Mills metric if and only if B is polystable.**

COROLLARY: Any tensor product of polystable bundles is polystable.

REMARK: This result **was generalized to coherent sheaves** by Bando and Siu.

Stability and transversally Kähler foliations

THEOREM: Let (M, Σ, ω_0) be a compact, complex manifold, $\dim M > 2$, and ω_0 a transversally Kähler, exact, equivariant form. Consider a vector bundle B with a Yang-Mills metric, $\deg_\omega B = 0$, and let ∇ denote the Yang-Mills connection. **Then ∇ is flat along the leaves of Σ, and equivariant with respect to $s \in \Gamma_M(\Sigma)$.**

REMARK: When Σ is quasiregular, **M is equipped with a holomorphic projection to the leaf space,** $\pi : M \longrightarrow X = M/\Sigma$. In this situation, the category of coherent sheaves can be described explicitly, in terms of a projective orbifold M/Σ.

THEOREM: Let F be a stable coherent sheaf on a be a compact, complex manifold (M, Σ, ω_0), with a transversally Kähler, exact form, $\dim M > 2$. Assume that Σ is quasiregular, and let $\pi : M \longrightarrow X = M/\Sigma$ be the projection map. **Then $F = \pi^* F_0$, for some coherent sheaf F_0 on M/Σ.**

COROLLARY: In these assumptions, **any coherent sheaf on M is filtrable**, that is, admits a filtration with rank 1 quotient sheaves.

REMARK: Filtrability is a very strong property! **It fails on almost all non-algebraic surfaces.**

Open questions

QUESTION: What happens when Σ is irregular?

Conjecture 1: Any coherent sheaf on (M, Σ, ω_0) is filtrable, even if Σ is irrecular.

Conjecture 2: Let M be a Vaisman manifold, obtained as a quotient $C(X)/\mathbb{Z}$ from a pair (X, φ), where X is Sasakian, and φ is a Sasakian automorphism. Consider the Lie group $G \subset \operatorname{Aut}(S)$ obtained as a closure of $\mathbb{Z} = \langle\varphi^n\rangle$. Then M is equipped with a natural projection $\pi : \ M \longrightarrow X/G$, with X/G parametrizing the closure of the appropriate leaves. Let (B, ∇) be a Hermitian vector bundle with connection which is flat on Σ and equivariant. **Then $(B, \nabla) = \pi^*(B_0, \nabla_0)$, for some (B_0, ∇_0) on X.**

REMARK: Conjecture 2 implies Conjecture 1. Also, Conjectures 1 and 2 are proven when $M = \mathbb{C}^n/\langle A\rangle$, where A is a linear map ("linear Hopf manifold").