Structure theorem for Vaisman manifolds

Lecture 2

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"Sasakian manifolds and locally conformally Kähler geometry"

Sasakian manifolds are Riemannian manifolds equipped with an $\mathbb{R}$-equivariant Kähler structure on their Riemannian cone. This allows one to characterize the Sasakian manifolds in terms of Kähler structure on the corresponding cones. Three lectures are planned.

1. A characterization of Sasakian manifolds in terms of CR-geometry (Nov. 03, 2009)

2. Locally conformally Kähler geometry, Vaisman manifolds and the structure theorem, providing an equivalence of Vaisman geometry and Sasakian geometry (Nov. 05, 2009)

3. A version of Kodaira theorem for Sasakian and Vaisman manifolds, giving a complex embedding of a Vaisman manifold into a Hopf manifold and a CR-embedding of a Sasakian manifold into a sphere (Nov. 06, 2009)

These results are obtained jointly with Liviu Ornea.

Contact manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and ω a symplectic form on $M \times \mathbb{R}^{>0}$. Suppose that ω is **homogeneous**: $\Psi_q^* \omega = q^2 \omega$, where $\Psi_q(m,t) = (m, qt)$. Then M is called **contact**.

Remark: The contact form on M is defined as $\theta = \omega \lrcorner \vec{T}$, where $\vec{T} = t\frac{d}{dt}$. Then $d\theta = [d, \cdot \lrcorner \vec{T}]\omega = \operatorname{Lie}_{\vec{T}} \omega = \omega$. Therefore, **the form $d\theta^{n-1} \wedge \theta = \frac{1}{n}\omega^n \lrcorner \vec{T}$ is non-degenerate on $M \times \{t_0\} \subset M \times \mathbb{R}^{>0}$.**

Remark: Usually, a contact manifold is defined as a **$(2n-1)$-manifold with 1-form θ such that $d\theta^{n-1} \wedge \theta$ is nowhere degenerate.**

Example: **An odd-dimensional sphere S^{2n-1} is contact.** Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n} \backslash 0$ has the standard symplectic form $\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n} \backslash 0$ has the standard symplectic form $\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry

Kähler manifolds.

Definition: Let (M, I) be a complex manifold, $\dim_{\mathbb{C}} M = n$, and g is Riemannian form. Then g is called **Hermitian** if $g(Ix, Iy) = g(x, y)$.

Remark: Since $I^2 = -\operatorname{Id}$, it is equivalent to $g(Ix, y) = -g(x, Iy)$. The form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

Definition: The differential form ω is called **the Hermitian form of (M, I, g)**.

Definition: A complex Hermitian manifold is called **Kähler** if $d\omega = 0$.

Remark: **Kähler manifolds are the main object of complex algebraic geometry** (algebraic geometry over $\mathbb{C}$). See e.g. Griffiths, Harris, *"Principles of Algebraic Geometry"*.

Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega\Big|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish (invariants of $U(n)$ are known since XIX century).

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.

Sasakian manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and (ω, I) a Kähler structure on $M \times \mathbb{R}^{>0}$. Suppose that ω is **homogeneous**: $\Psi_q^*\omega = q^2 g$, where $\Psi_q(m,t) = (m, qt)$, and I is Ψ_q-invariant. Then M is called **Sasakian**, and $M \times \mathbb{R}^{>0}$ its **Kähler cone**.

Sasakian geometry is an odd-dimensional counterpart to Kähler geometry

Remark: A Sasakian manifold is obviosly contact. Indeed, **a Sasakian manifold is a contact manifold equipped with a compatible Riemannian metric.**

Example: **An odd-dimensional sphere S^{2n-1} is Sasakian.** Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{C}^n \backslash 0$ has the standard Kähler form $\sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$ which is obviously homogeneous.

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.

Quasiregular Sasakian manifolds.

Definition: Given a contact manifold (M, θ) with a Riemannian structure g, the dual vector field $\theta^\sharp$ is called **the Reeb field** of (M, θ, g).

Remark: For any Sasakian manifold, **the Reeb field generates a flow of diffeomorphisms acting on M by contact isometries**. This is obvious from the definition, because the Reeb field $\theta^\sharp = It\frac{d}{dt}$ acts by holomorphic isometries on the Kähler cone.

Definition: A Sasakian manifold M is called **quasiregular** if all orbits of the Reeb flow are compact. The space of orbits of the Reeb flow is a complex orbifold. **Every quasiregular Sasakian manifold is a total space of S^1-bundle over a complex orbifold.**

This is easy to see, because the quotient of M over the Reeb flow is the same as the quotient of CM over its complexification, generated by $\theta^\sharp$ and $I\theta^\sharp$.

Examples of Sasakian manifolds.

Example: Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $CX \subset \mathbb{C}^{n+1}\backslash 0$ the corresponding cone. The cone CX is obviously Kähler and homogeneous, hence **the intersection $CX \cap S^{2n-1}$ is Sasakian.** This intersection is an S^1-bundle over X. This construction gives many interesting contact manifolds, including Milnor's exotic 7-spheres, which happen to be Sasakian.

REMARK: In other words, **a link of a homogeneous singularity is always Sasakian.**

REMARK: Every quasiregular Sasakian manifold is obtained this way, for some Kähler metric on $\mathbb{C}^{n+1}$ (Ornea-V., arXiv:math/0609617).

REMARK: All 3-dimensional Sasakian manifolds are quasiregular (H. Geiges, 1997, F. Belgun, 2000).

REMARK: Every Sasakian manifold is diffeomorphic to a quasiregular one (Ornea-V., arXiv:math/0306077)

Sasakian manifolds and circle bundles

DEFINITION: Let S be a projective manifold (or orbifold), L a holomorphic Hermitian line bundle, $\omega \in \Lambda^{1,1}(S)$ its curvature. If $\omega > 0$, the bundle is called **positive**.

REMARK: Consider the function $v(l) = |l|^2$ on $\tilde{M} := \operatorname{Tot}(L^* \backslash 0)$. **Then $dd^c(v)$ is a Kähler form on $\tilde{M}$.**

EXAMPLE: Let X be a total space of the unit circle bundle in L^*. **Then $\tilde{M} = C(X)$ is its Riemannian cone.** In particular, **X is Sasakian.**

REMARK: Every quasiregular Sasakian manifold X **is a circle bundle over its quotient X/S^1, which is a Kähler orbifold.**

Locally conformally Kähler manifolds

REMARK: It is **always assumed** that LCK manifolds have $\dim_{\mathbb{C}} > 1$.

DEFINITION: Let M be a complex manifold, $\tilde M \longrightarrow M$ its covering, $\tilde M/\Gamma = M$, and Γ the **monodromy group** freely acting on $\tilde M$. Assume that $\tilde M$ is Kähler, and Γ acts on $\tilde M$ by homotheties. Then M is called **locally conformally Kähler** (LCK).

EXAMPLE: A Hopf manifold $\mathbb{C}^n\backslash 0/\langle A\rangle$ is obviously LCK, if $A(x) = qx$, $|q| > 1$.

EXAMPLE: Let X be a Sasakian manifold, and $\tilde M = C(X) = X \times \mathbb{R}^{>0}$ its Kähler cone, equipped with an action by $\mathbb{Z} = \left\langle (x,t) \mapsto (x,qt)\right\rangle$, $q > 1$. **Then $C(X)/\mathbb{Z}$ is LCK.**

DEFINITION: Let M be an LCK manifold, $(\tilde M, \tilde\omega)$ its covering. An **LCK metric** on M is a metric conformal to $\tilde\omega$.

REMARK: If $\omega = f\tilde\omega$, one has $d\omega = df \wedge \omega'$. Therefore, **an LCK metric satisfies $d\omega = \theta\wedge\omega$,** for some closed 1-form θ.

REMARK: If $\theta = df$, the form $e^{-f}\omega$ is Kähler.

LCK manifolds and Hermitian geometry

CLAIM: If (M, I, ω) is a Hermitian manifold with $d\omega = \theta \wedge \omega$, for some closed 1-form θ, **then (M, I) is LCK.**

Proof: For some covering of M the lift of θ is exact, and there **the lift of ω is conformal to a Kähler form.** ■

DEFINITION: A Hermitian manifold (M, I, ω) is called **LCK** if $d\omega = \theta \wedge \omega$, for some closed 1-form θ.

DEFINITION: The form θ is called **the Lee form** of an LCK-manifold. The dual vector field $\theta^\sharp$ – **the Lee field**

CLAIM: If $\dim_{\mathbb{C}} M > 2$, and $d\omega = \theta \wedge \omega$, **then θ is always exact.**

Proof: $0 = d^2\omega = d\theta \wedge \omega$, but $\cdot \wedge \omega : \Lambda^2(M) \longrightarrow \Lambda^4(M)$ **is always injective for** $\dim_{\mathbb{C}} M > 2$. ■

Vaisman manifolds

DEFINITION: An LCK manifold (M,ω,θ) is called **Vaisman** ("generalized Hopf") if $\nabla_{LC}\theta = 0$, where ∇_{LC} is the Levi-Civita connection.

THEOREM: (Kamishima-Ornea, 2001) If M is compact, **this is equivalent to M admitting a conformal holomorphic flow,** acting non-isometrically on its Kähler covering.

EXAMPLE: A Hopf manifold $\mathbb{C}^n\backslash 0/\langle A\rangle$ is Vaisman. It is isometric to $S^{2n-1}\times S^1$, and the Lee field is $\frac{d}{dt}$.

EXAMPLE: Let X be a Sasakian manifold, and $C(X)$ its Kähler cone, equipped with an action by $\mathbb{Z} = \left\langle (x,t)\mapsto (x,qt)\right\rangle$, $q>1$. **Then $C(X)/\mathbb{Z}$ is Vaisman.** Indeed, $C(X)/\mathbb{Z}$ is isometric to $X\times S^1$, with the Lee field is $\frac{d}{dt}$.

EXAMPLE: The same happens if $\mathbb{Z} = \left\langle (x,t)\mapsto (\varphi(x),qt)\right\rangle$, $q>1$, and φ is a Sasakian automorphism of X.

Structure theorem for Vaisman manifolds

THEOREM: **Every Vaisman manifold is obtained as $C(X)/\mathbb{Z}$,** where X is Sasakian, $\mathbb{Z} = \left\langle (x,t) \mapsto (\varphi(x), qt) \right\rangle$, $q > 1$, and φ is a Sasakian automorphism of X. Moreover, the triple (X, φ, q) is unique.

REMARK: This is equivalent to the **existence of a Riemannian submersion $M \longrightarrow S^1$,** with Sasakian fibers.

Proof. Step 1: Since $\theta^\sharp$ is parallel and Killing, $M = X \times \mathbb{R}$ locally. Fix $x_0 \in M$. Then the projection $M = X \times \mathbb{R}$ to $\mathbb{R}$ is induced by $x \longrightarrow \int_{\gamma_{x_0,x}} \theta$, for $\gamma_{x_0,x}$ some path connecting x and x_0. Therefore, $M = X \times \mathbb{R}$ whenever θ is exact.

DEFINITION: A **monodromy group** $\operatorname{Mon}(M)$ of an LCK manifold M is the smallest group Γ such that $M = \tilde{M}/\Gamma$ and $\tilde{M}$ is Kähler.

REMARK: **This is equivalent to the lift of θ being exact**.

The proof of Structure theorem for Vaisman manifolds

Proof. Step 2: Let $\gamma_1, ..., \gamma_k \in H_1(M, \mathbb{Z})$ be generators of homology, and $\alpha_i \int_{\gamma_i} \theta$ the corresponding periods. One has a map $M \longrightarrow \mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle$, with a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & \mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle \end{array}$$

with vertical lines $x \longrightarrow \int_{\gamma_{x_0,x}} \theta$. **The Riemannian submersion to S^1 will be obtained if $\mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle = S^1$.**

Step 3: Let $G \subset \pi_1(M)$ be the group generated by all $\gamma \in \pi_1(M)$ such that $\int_\gamma \theta = 0$. **Then $\Gamma = \pi_1(M)/G$ is the monodromy group of M. Therefore, $\mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle = S^1 \iff \operatorname{Mon}(M) = \mathbb{Z}$.**

Computation of the monodromy group of a Vaisman manifold

DEFINITION: Let G be a group of holomorphic automorphisms of M obtained as a closure of the Lee field action.

REMARK: Since $\theta^\sharp$ acts by isometries, G is a compact torus $G = (S^1)^k$.

CLAIM: Consider the group $\tilde G$ of pairs $\tilde f \in \operatorname{Aut}(\tilde M)$, $f \in G$, making the following diagram commutative.

$$\begin{array}{ccc} \tilde M & \xrightarrow{\tilde f} & \tilde M \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

Then $\tilde G \cong (S^1)^{k-1} \times \mathbb{R}$.

REMARK: From this claim, the isomorphism $\operatorname{Mon}(M) = \mathbb{Z}$ follows immediately. Indeed, $\operatorname{Mon}(M) \subset \ker p : \tilde G \longrightarrow G$.

Computation of the monodromy group of a Vaisman manifold (part 2)

Proof of $\tilde{G} \cong (S^1)^{k-1} \times \mathbb{R}$.

Step 1: $\tilde{G}$ is a covering of G, and **the kernel of this projection is** $\operatorname{Mon}(M)$**.**

Step 2: Let $\tilde{G}_0 \subset \tilde{G}$ be a subgroup acting on $\tilde{M}$ by isometries. Since $\tilde{G}$ acts on $\tilde{M}$ by homotheties, $\tilde{G}_0$ **has codimension 1**. Therefore, $\tilde{G}_0$ cannot intersect $\operatorname{Mon}(M)$ and **it maps injectively to** $\operatorname{Aut}(M) \cong (S^1)^k$**.**

Step 3: We obtain that $\tilde{G}_0 \cong (S^1)^{k-1}$ (it's codimension 1).

Step 4: Since $\tilde{G}_0$ meets every component of $\tilde{G}$, it is connected. **Therefore,** $\tilde{G} \cong \tilde{G}_0 \times \mathbb{R} \cong (S^1)^{k-1} \times \mathbb{R}$**.** ∎