Structure theorem for Vaisman manifolds

Lecture 2

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"Sasakian manifolds and locally conformally Kähler geometry"

Sasakian manifolds are Riemannian manifolds equipped with an $\mathbb{R}$-equivariant Kähler structure on their Riemannian cone. This allows one to characterize the Sasakian manifolds in terms of Kähler structure on the corresponding cones. Three lectures are planned.

1. A characterization of Sasakian manifolds in terms of CR-geometry (Nov. 03, 2009)

2. Locally conformally Kähler geometry, Vaisman manifolds and the structure theorem, providing an equivalence of Vaisman geometry and Sasakian geometry (Nov. 05, 2009)

3. A version of Kodaira theorem for Sasakian and Vaisman manifolds, giving a complex embedding of a Vaisman manifold into a Hopf manifold and a CR-embedding of a Sasakian manifold into a sphere (Nov. 06, 2009)

These results are obtained jointly with Liviu Ornea.
Contact manifolds.

**Definition:** Let $M$ be a smooth manifold, $\dim M = 2n - 1$, and $\omega$ a symplectic form on $M \times \mathbb{R}^{>0}$. Suppose that $\omega$ is **homogeneous:** $\Psi^*\omega = q^2\omega$, where $\Psi_q(m, t) = (m, qt)$. Then $M$ is called **contact**.

**Remark:** The contact form on $M$ is defined as $\theta = \omega \downarrow \vec{T}$, where $\vec{T} = t \frac{d}{dt}$. Then $d\theta = [d, \downarrow \vec{T}]\omega = \text{Lie}_{\vec{T}}\omega = \omega$. Therefore, the form $d\theta^{n-1} \wedge \theta = \frac{1}{n} \omega^n \downarrow \vec{T}$ is non-degenerate on $M \times \{t_0\} \subset M \times \mathbb{R}^{>0}$.

**Remark:** Usually, a contact manifold is defined as a $(2n - 1)$-manifold with 1-form $\theta$ such that $d\theta^{n-1} \wedge \theta$ is nowhere degenerate.

**Example:** An odd-dimensional sphere $S^{2n-1}$ is contact. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n} \setminus 0$ has the standard symplectic form $\sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n} \setminus 0$ has the standard symplectic form $\sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry.
Kähler manifolds.

**Definition:** Let \((M, I)\) be a complex manifold, \(\dim_{\mathbb{C}} M = n\), and \(g\) is Riemannian form. Then \(g\) is called **Hermitian** if \(g(Ix, Iy) = g(x, y)\).

**Remark:** Since \(I^2 = -\text{Id}\), it is equivalent to \(g(Ix, y) = -g(x, Iy)\). The form \(\omega(x, y) := g(x, Iy)\) is skew-symmetric.

**Definition:** The differential form \(\omega\) is called the **Hermitian form of** \((M, I, g)\).

**Definition:** A complex Hermitian manifold is called **Kähler** if \(d\omega = 0\).

**Remark:** Kähler manifolds are the main object of complex algebraic geometry (algebraic geometry over \(\mathbb{C}\)). See e.g. Griffiths, Harris, “Principles of Algebraic Geometry”.
Examples of Kähler manifolds.

**Definition:** Let $M = CP^n$ be a complex projective space, and $g$ a $U(n + 1)$-invariant Riemannian form. It is called **Fubini-Study form on $CP^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n + 1)$.

**Remark:** For any $x \in CP^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_xCP^n$ is $U(n)$-invariant, hence unique up to a constant.

**Claim:** **Fubini-Study form is Kähler.** Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $C^n$, but such a form must vanish (invariants of $U(n)$ are known since XIX century).

**Corollary:** Every projective manifold (complex submanifold of $CP^n$) is Kähler.
**Sasakian manifolds.**

**Definition:** Let $M$ be a smooth manifold, $\dim M = 2n - 1$, and $(\omega, I)$ a Kähler structure on $M \times \mathbb{R}^\times$. Suppose that $\omega$ is \textbf{homogeneous}: $\Psi^\ast \omega = q^2 g$, where $\Psi_q(m, t) = (m, qt)$, and $I$ is $\Psi_q$-invariant. Then $M$ is called \textbf{Sasakian}, and $M \times \mathbb{R}^\times$ its \textbf{Kähler cone}.

\textbf{Sasakian geometry is an odd-dimensional counterpart to Kähler geometry}

**Remark:** A Sasakian manifold is obviously contact. Indeed, a Sasakian manifold is a contact manifold equipped with a compatible Riemannian metric.

**Example:** An odd-dimensional sphere $S^{2n-1}$ is Sasakian. Indeed, its cone $S^{2n-1} \times \mathbb{R}^\times = \mathbb{C}^n \backslash 0$ has the standard Kähler form $\sqrt{-1} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ which is obviously homogeneous.

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.
Quasiregular Sasakian manifolds.

Definition: Given a contact manifold \((M, \theta)\) with a Riemannian structure \(g\), the dual vector field \(\theta^\#\) is called the Reeb field of \((M, \theta, g)\).

Remark: For any Sasakian manifold, the Reeb field generates a flow of diffeomorphisms acting on \(M\) by contact isometries. This is obvious from the definition, because the Reeb field \(\theta^\# = It\frac{d}{dt}\) acts by holomorphic isometries on the Kähler cone.

Definition: A Sasakian manifold \(M\) is called quasiregular if all orbits of the Reeb flow are compact. The space of orbits of the Reeb flow is a complex orbifold. Every quasiregular Sasakian manifold is a total space of \(S^1\)-bundle over a complex orbifold.

This is easy to see, because the quotient of \(M\) over the Reeb flow is the same as the quotient of \(CM\) over its complexification, generated by \(\theta^\#\) and \(I\theta^\#\).
Examples of Sasakian manifolds.

Example: Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $CX \subset \mathbb{C}^{n+1}\setminus 0$ the corresponding cone. The cone $CX$ is obviously Kähler and homogeneous, hence the intersection $CX \cap S^{2n-1}$ is Sasakian. This intersection is an $S^1$-bundle over $X$. This construction gives many interesting contact manifolds, including Milnor’s exotic 7-spheres, which happen to be Sasakian.

REMARK: In other words, a link of a homogeneous singularity is always Sasakian.

REMARK: Every quasiregular Sasakian manifold is obtained this way, for some Kähler metric on $\mathbb{C}^{n+1}$ (Ornea-V., arXiv:math/0609617).

REMARK: All 3-dimensional Sasakian manifolds are quasiregular (H. Geiges, 1997, F. Belgun, 2000).

REMARK: Every Sasakian manifold is diffeomorphic to a quasiregular one (Ornea-V., arXiv:math/0306077).
Sasakian manifolds and circle bundles

**DEFINITION:** Let $S$ be a projective manifold (or orbifold), $L$ a holomorphic Hermitian line bundle, $\omega \in \Lambda^{1,1}(S)$ its curvature. If $\omega > 0$, the bundle is called positive.

**REMARK:** Consider the function $v(l) = |l|^2$ on $\tilde{M} := \text{Tot}(L^* \setminus 0)$. Then $dd^c(v)$ is a Kähler form on $\tilde{M}$.

**EXAMPLE:** Let $X$ be a total space of the unit circle bundle in $L^*$. Then $\tilde{M} = C(X)$ is its Riemannian cone. In particular, $X$ is Sasakian.

**REMARK:** Every quasiregular Sasakian manifold $X$ is a circle bundle over its quotient $X/S^1$, which is a Kähler orbifold.
Locally conformally Kähler manifolds

**REMARK:** It is **always assumed** that LCK manifolds have \( \dim_\mathbb{C} > 1 \).

**DEFINITION:** Let \( M \) be a complex manifold, \( \tilde{M} \rightarrow M \) its covering, \( \tilde{M}/\Gamma = M \), and \( \Gamma \) the monodromy group freely acting on \( \tilde{M} \). Assume that \( \tilde{M} \) is Kähler, and \( \Gamma \) acts on \( \tilde{M} \) by homotheties. Then \( M \) is called **locally conformally Kähler** (LCK).

**EXAMPLE:** A Hopf manifold \( \mathbb{C}^n \backslash 0 / \langle A \rangle \) is obviously LCK, if \( A(x) = qx, \ |q| > 1 \).

**EXAMPLE:** Let \( X \) be a Sasakian manifold, and \( \tilde{M} = C(X) = X \times \mathbb{R}^0 \) its Kähler cone, equipped with an action by \( \mathbb{Z} = \langle (x, t) \mapsto (x, qt) \rangle, \ q > 1 \). Then \( C(X)/\mathbb{Z} \) is LCK.

**DEFINITION:** Let \( M \) be an LCK manifold, \( (\tilde{M}, \tilde{\omega}) \) its covering. An **LCK metric** on \( M \) is a metric conformal to \( \tilde{\omega} \).

**REMARK:** If \( \omega = f \tilde{\omega} \), one has \( d\omega = df \wedge \omega' \). Therefore, an LCK metric satisfies \( d\omega = \theta \wedge \omega \), for some closed 1-form \( \theta \).

**REMARK:** If \( \theta = df \), the form \( e^{-f}\omega \) is Kähler.
LCK manifolds and Hermitian geometry

**CLAIM:** If $(M, I, \omega)$ is a Hermitian manifold with $d\omega = \theta \wedge \omega$, for some closed 1-form $\theta$, then $(M, I)$ is LCK.

**Proof:** For some covering of $M$ the lift of $\theta$ is exact, and there the lift of $\omega$ is conformal to a Kähler form. ■

**DEFINITION:** A Hermitian manifold $(M, I, \omega)$ is called **LCK** if $d\omega = \theta \wedge \omega$, for some closed 1-form $\theta$.

**DEFINITION:** The form $\theta$ is called the **Lee form** of an LCK-manifold. The dual vector field $\theta^\sharp$ – the **Lee field**

**CLAIM:** If $\dim \mathbb{C} M > 2$, and $d\omega = \theta \wedge \omega$, then $\theta$ is always exact.

**Proof:** $0 = d^2 \omega = d(\theta \wedge \omega)$, but $\cdot \wedge : \Lambda^2(M) \to \Lambda^4(M)$ is always injective for $\dim \mathbb{C} M > 2$. ■
Structure theorem for Vaisman manifolds

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Vaisman manifolds

**DEFINITION:** An LCK manifold \((M, \omega, \theta)\) is called **Vaisman** (“generalized Hopf”) if \(\nabla_{LC} \theta = 0\), where \(\nabla_{LC}\) is the Levi-Civita connection.

**THEOREM:** (Kamishima-Ornea, 2001) If \(M\) is compact, **this is equivalent to** \(M\) admitting a conformal holomorphic flow, acting non-isometrically on its Kähler covering.

**EXAMPLE:** A Hopf manifold \(\mathbb{C}^n \setminus 0 / \langle A \rangle\) is Vaisman. It is isometric to \(S^{2n-1} \times S^1\), and the Lee field is \(\frac{d}{dt}\).

**EXAMPLE:** Let \(X\) be a Sasakian manifold, and \(C(X)\) its Kähler cone, equipped with an action by \(\mathbb{Z} = \left\langle (x, t) \mapsto (x, qt) \right\rangle\), \(q > 1\). **Then** \(C(X)/\mathbb{Z}\) is Vaisman. Indeed, \(C(X)/\mathbb{Z}\) is isometric to \(X \times S^1\), with the Lee field is \(\frac{d}{dt}\).

**EXAMPLE:** The same happens if \(\mathbb{Z} = \left\langle (x, t) \mapsto (\varphi(x), qt) \right\rangle\), \(q > 1\), and \(\varphi\) is a Sasakian automorphism of \(X\).
Structure theorem for Vaisman manifolds

**THEOREM:** Every Vaisman manifold is obtained as $C(X)/\mathbb{Z}$, where $X$ is Sasakian, $\mathbb{Z} = \left\langle (x, t) \mapsto (\varphi(x), qt) \right\rangle$, $q > 1$, and $\varphi$ is a Sasakian automorphism of $X$. Moreover, the triple $(X, \varphi, q)$ is unique.

**REMARK:** This is equivalent to the existence of a Riemannian submersion $M \rightarrow S^1$, with Sasakian fibers.

**Proof. Step 1:** Since $\theta$ is parallel and Killing, $M = X \times \mathbb{R}$ locally. Fix $x_0 \in M$. Then the projection $M = X \times \mathbb{R}$ to $\mathbb{R}$ is induced by $x \mapsto \int_{\gamma_{x_0,x}} \theta$, for $\gamma_{x_0,x}$ some path connecting $x$ and $x_0$. Therefore, $M = X \times \mathbb{R}$ whenever $\theta$ is exact.

**DEFINITION:** A **monodromy group** $\text{Mon}(M)$ of an LCK manifold $M$ is the smallest group $\Gamma$ such that $M = \tilde{M}/\Gamma$ and $\tilde{M}$ is Kähler.

**REMARK:** This is equivalent to the lift of $\theta$ being exact.
The proof of Structure theorem for Vaisman manifolds

Proof. Step 2: Let $\gamma_1, ..., \gamma_k \in H_1(M, \mathbb{Z})$ be generators of homology, and $\alpha_i \int_{\gamma_i} \theta$ the corresponding periods. One has a map $M \rightarrow \mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle$, with a commutative diagram

\[
\begin{array}{ccc}
\tilde{M} & \rightarrow & M \\
\downarrow & & \downarrow \\
\mathbb{R} & \rightarrow & \mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle
\end{array}
\]

with vertical lines $x \rightarrow \int_{\gamma, x_0, x} \theta$. The Riemannian submersion to $S^1$ will be obtained if $\mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle = S^1$.

Step 3: Let $G \subset \pi_1(M)$ be the group generated by all $\gamma \in \pi_1(M)$ such that $\int_{\gamma} \theta = 0$. Then $\Gamma = \pi_1(M)/G$ is the monodromy group of $M$. Therefore, $\mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle = S^1 \iff \text{Mon}(M) = \mathbb{Z}$. 
Computation of the monodromy group of a Vaisman manifold

**DEFINITION:** Let $G$ be a group of holomorphic automorphisms of $M$ obtained as a closure of the Lee field action.

**REMARK:** Since $\theta^{\sharp}$ acts by isometries, $G$ is a compact torus $G = (S^1)^k$.

**CLAIM:** Consider the group $\tilde{G}$ of pairs $\tilde{f} \in \text{Aut}(\tilde{M}), f \in G$, making the following diagram commutative.

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M} \\
\downarrow{\pi} & & \downarrow{\pi} \\
M & \xrightarrow{f} & M
\end{array}
\]

Then $\tilde{G} \cong (S^1)^{k-1} \times \mathbb{R}$.

**REMARK:** From this claim, the isomorphism $\text{Mon}(M) = \mathbb{Z}$ follows immediately. Indeed, $\text{Mon}(M) \subset \ker p : \tilde{G} \longrightarrow G$. 
Structure theorem for Vaisman manifolds

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Computation of the monodromy group of a Vaisman manifold (part 2)

Proof of \( \tilde{G} \cong (S^1)^{k-1} \times \mathbb{R} \).

Step 1: \( \tilde{G} \) is a covering of \( G \), and the kernel of this projection is \( \text{Mon}(M) \).

Step 2: Let \( \tilde{G}_0 \subset \tilde{G} \) be a subgroup acting on \( \tilde{M} \) by isometries. Since \( \tilde{G} \) acts on \( \tilde{M} \) by homotheties, \( \tilde{G}_0 \) has codimension 1. Therefore, \( \tilde{G}_0 \) cannot intersect \( \text{Mon}(M) \) and it maps injectively to \( \text{Aut}(M) \cong (S^1)^k \).

Step 3: We obtain that \( \tilde{G}_0 \cong (S^1)^{k-1} \) (it’s codimension 1).

Step 4: Since \( \tilde{G}_0 \) meets every component of \( \tilde{G} \), it is connected. Therefore, \( \tilde{G} \cong \tilde{G}_0 \times \mathbb{R} \cong (S^1)^{k-1} \times \mathbb{R} \). ■