Hypercomplex Manifolds

Quaternionic geometry: an introduction

Isometries of $\mathbb{R}^2$ *are expressed in terms of complex numbers. This allows one to answer geometry questions algebraically.*

QUESTION: Can we do that in dimension 3?

ANSWER: Yes!

Sir William Rowan Hamilton
(August 4, 1805 – September 2, 1865)

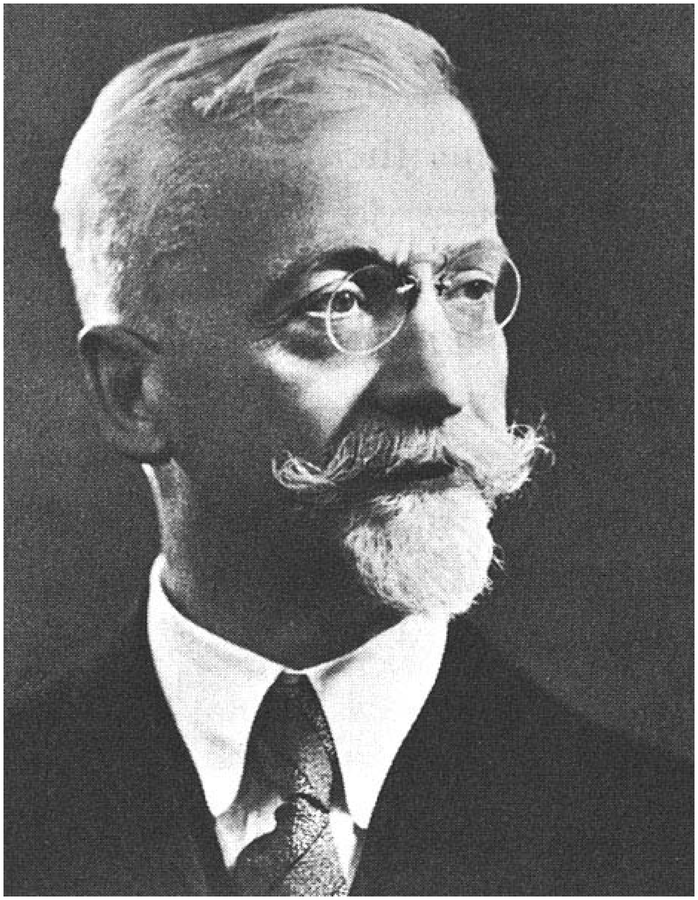
Broom Bridge

"Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

$$I^2 = J^2 = K^2 = IJK = -1$$

and cut it on a stone of this bridge."

Fast forward 70 years.

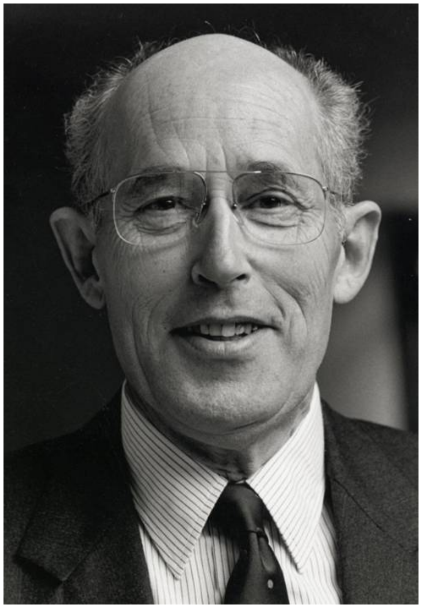


Élie Joseph Cartan
(9 April 1869 – 6 May 1951)

"Quaternionic structures" in the sense of Elie Cartan **don't exist.**

THEOREM: Let $f: \mathbb{H}^n \longrightarrow H^m$ be a function, defined locally in some open subset of n-dimensional quaternion space $\mathbb{H}^n$. Suppose that the differential Df is $\mathbb{H}$-linear. Then f is a linear map.

Proof (a modern one): The graph of f is "hypercomplex manifold" in $\mathbb{H}^n \times H^m$, hence "geodesically complete", hence linear. ■



Marcel Berger

Classification of holonomies.

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

In Riemannian Geometry, we have two kinds of quaternionic manifolds:

quaternionic-Kähler, holonomy $Sp(n)Sp(1)$

hyperkähler manifolds , holonomy $Sp(n)$

The quaternionic-Kähler manifolds (name is misleading) are rare: all compact examples are symmetric. Compact hyperkähler manifolds are constructed by Yau's solution of Calabi's conjecture.

Nota bene: $Sp(n)$ is a group of matrices $h \in GL(n, \mathbb{H})$ preserving the quaternionic Hermitian structure.

What is Kähler geometry?

1. Kähler manifold is a Riemannian manifold with holonomy $U(n)$

2. This means that we are given $I: TM \longrightarrow TM$, $I^2 = -\operatorname{Id}$, and $\nabla I = 0$. Also, I is orthogonal: $g(Ix, Iy) = g(x, y)$.

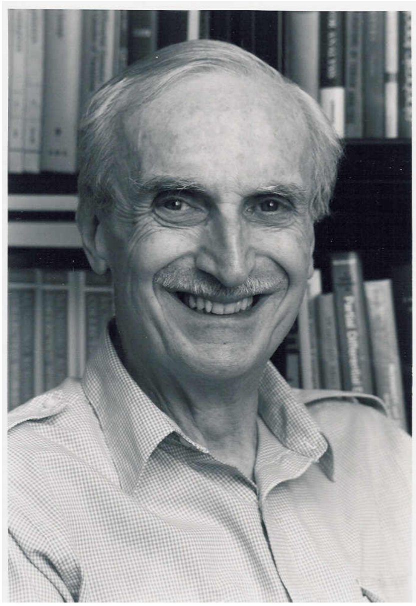
3. The operator I is a complex structure on M. The metric g is Hermitian, hence the form $\omega(x, y) := G(Ix, y)$ is skew-symmetric ("Hermitian form"). Also $\nabla\omega = 0$, hence ω is closed.

4. These data are used to provide an equivalent definition (which becomes a theorem now).

THEOREM: Let M be a complex, Hermitian manifold, ω its Hermitian form. Then (M, I, ω) is Kähler if and only if $d\omega = 0$.

Kähler manifolds are ***bona fide*** **symplectic.**

Algebraic complex manifolds manifolds are always Kähler



Eugenio Calabi

Definition: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M,I,J,K,g) is called **hyperkähler**.

Holomorphic symplectic geometry

A hyperkähler manifold (M, I, J, K), considered as a complex manifold (M, I), is holomorphically symplectic (equipped with a holomorphic, non-degenerate 2-form). Recall that, M is equipped with 3 symplectic forms ω_I, ω_J, ω_K.

LEMMA: The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic 2-form on (M, I). ■

Converse is also true, as follows from the famous conjecture, made by Calabi in 1952.

THEOREM: (S.-T. Yau, 1978) Let M be a compact, holomorphically symplectic Kähler manifold. Then M admits a hyperkähler metric, which is uniquely determined by the cohomology class of its Kähler form ω_I.

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

Hyperkähler geometry

0. **Induced complex structures** are complex structures of form

$$L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.$$

They are non-algebraic (mostly). Indeed, for generic a, b, c, (M, L) has no divisors.

1. These complex structures can be glued together to form a "twistor space", $\operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$. The hyperkähler structure can be defined in terms of a twistor space. You can have "hyperkähler singular spaces", and even schemes.

2. **Trianalytic subvarieties** are closed subsets which are complex analytic with respect to I, J, K.

- Let L be a generic induced complex structure. Then all complex subvarieties of (M, L) are trianalytic.

- A normalization of a trianalytic subvariety is smooth and hyperkähler.

- A complex deformation of a trianalytic subvariety is again trianalytic, the corresponding moduli space is hyperkähler.

3. Similar results (also very strong) are true for vector bundles which are holomorphic under I, J, K ("hyperholomorphic bundles")

Another world: HYPERCOMPLEX MANIFOLDS

Definition: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\mathrm{Id}\,.$$

Suppose that I, J, K are integrable. Then (M, I, J, K) is called **a hypercomplex manifold**.

0. Obata connection.

1. In dimension 1, we have a classification, due to Charles P. Boyer.

2. In dimension $i > 1$, no classification is known (and no conjectures either).

3. Many examples, due to D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen.

QUESTIONS

1. Given a complex manifold M, when M admits a hypercomplex structure? How many?

2. Can we classify the affine hypercomplex manifolds (one where the Obata connection is flat)?

3. What are possible holonomies of Obata connection?

4. What are the trianalytic subvarieties and automorphisms?

THEOREM: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that (M, I) admits a Kähler structure. Then M is hyperkähler.