# Rational curves on non-Kähler manifolds

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## Plan

1. Non-Kähler manifolds. Twistor spaces.

2. Different Hermitian structures and conditions.

3. Rational curves on twistor spaces and special metrics.

4. Applications.

## Non-Kähler manifolds

Constructions of compact, non-Kähler manifolds.

1. **Locally conformally Kähler manifolds.** A quotient of a Kähler manifold by a discrete group acting by homotheties. **Never admits a Kähler metric.** (Vaisman).

**Example:** Hopf manifold $(\mathbb{C}^n \backslash 0)/\mathbb{Z}$.

2. **Left-invariant complex structures** on Lie groups or their quotients by discrete groups. **Almost never Kähler** (exception: a torus).

3. **Twistor spaces**. Huge. **Any finite-generated group can be a fundamental of a twistor space** (Taubes; Panov-Petrunin). **Never Kähler** (Hitchin), except $\mathbb{C}P^3$ and flag spaces.

4. Moishezon (and Fujiki class C) manifolds.

## Twistor spaces (hyperkähler geometry)

**DEFINITION:** A **hyperkähler structure** on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**DEFINITION: Induced complex structures** on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

**DEFINITION:** A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on $M$ induced by $J \in S^2 \subset \mathbb{H}$. Let $I_J$ denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata)

**EXAMPLE:** If $M = \mathbb{H}^n$, $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$ (total space of a vector bundle $(\mathcal{O}(1)^{\oplus n})$.

**REMARK: For $M$ compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.**

## Twistor spaces (4-manifolds)

**DEFINITION:** Let $M$ be a Riemannian 4-manifold. Consider the action of the Hodge $*$-operator: $* : \Lambda^2 M \longrightarrow \Lambda^2 M$. Since $*^2 = 1$, the eigenvalues are $\pm 1$, and one has a decomposition $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$ onto **autodual** ($*\eta = \eta$) and **anti-autodual** ($*\eta = -\eta$) forms.

**REMARK:** If one changes the orientation of $M$, leaving metric the same, $\Lambda^+ M$ and $\Lambda^- M$ are exchanged. **Therefore,** $\dim \Lambda^2 M = 6$ **implies** $\dim \Lambda^\pm(M) = 3$**.**

**REMARK:** Using the isomorphism $\Lambda^2 M = \mathfrak{so}(TM)$, we interpret $\eta \in \Lambda^2_m M$ as an endomorphisms of $T_m M$. **Then the unit vectors $\eta \in \Lambda^+_m M$ correspond to oriented, orthogonal complex structures on $T_m M$.**

**DEFINITION:** Let $\operatorname{Tw}(M) := S\Lambda^+ M$ be the set of unit vectors in $\Lambda^+ M$. At each point $(m, s) \in \operatorname{Tw}(M)$, consider the decoposition $T_{m,s} \operatorname{Tw}(M) = T_m M \oplus T_s S\Lambda^+_m M$, induced by the Levi-Civita connection. Let $I_s$ be the complex structure on $T_m M$ induced by $s$, $I_{S\Lambda^+_m M}$ the complex structure on $S\Lambda^+_m M = S^2$ induced by the metrics and orientation, and $\mathcal{I} : T_{m,s} \operatorname{Tw}(M) \longrightarrow T_{m,s} \operatorname{Tw}(M)$ be equal to $\mathcal{I}_s \oplus I_{S\Lambda^+_m M}$. An almost complex manifold $(\operatorname{Tw}(M), \mathcal{I})$ is called **the twistor space** of $M$.

## PROPERTIES OF TWISTOR SPACES:

1. The almost complex structure on $\operatorname{Tw}(M), \mathcal{I}$ **is a conformal invariant** of $M$. Moreover, **one can reconstruct the conformal structure** on $M$ from the almost complex structure on $\operatorname{Tw}(M)$ and its anticomplex involution $(m,s) \longrightarrow (m,-s)$.

2. **$\operatorname{Tw}(M), \mathcal{I}$ is a complex manifold if and only if $W^+ = 0$,** where $W^+$ ("self-dual conformal curvature") is an autodual component of the curvature tensor. Such manifolds are called **conformally half-flat** or **ASD (anti-selfdual).**

3. For a hyperkähler 4-fold, two definitions of twistor spaces coincide.

**Rational curves on** $\operatorname{Tw}(M)$**.**

**DEFINITION: An ample rational curve** on a complex manifold $M$ is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$, with $i_k > 0$. It is called **a quasi-line** if all $i_k = 1$.

**CLAIM:** Let $M$ be a compact complex manifold containing a an ample rational line. **Then any $N$ points $z_1, ..., z_N$ can be connected by an ample rational curve.**

**CLAIM:** Let $M$ be a Riemannian 4-manifold, $\operatorname{Tw}(M) \xrightarrow{\sigma} M$ its twistor space, $m \in M$ a point, and $S_m := \sigma^{-1}(m) = S\Lambda^+_m(M)$ the corresponding $S^2$ in $\operatorname{Tw}(M)$. **Then $S_m$ is a quasi-line.**

**Proof:** Since the claim is essentially infinitesimal, it suffices to check it when $M$ is flat. **Then $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus 2}) \cong \mathbb{C}P^3 \backslash \mathbb{C}P^1$, and $S_m$ is a section of $\mathcal{O}(1)^{\oplus 2}$.** ■

## Hermitian structures on complex manifolds.

Let $(M,I,g)$ be a complex Hermitian $n$-manifold, $\omega \in \Lambda^{1,1}(M)$ its Hermitian form.

**REMARK:** For each $1 \leqslant k \leqslant n-1$, the condition $d(\omega^k)=0$ implies $d\omega = 0$. Hermitian metric is called **balanced** if $d(\omega^{n-1})=0$. All twistor spaces are balanced (Hitchin). All Moishezon manifolds are balanced (Alessandrini-Bassaneli).

**REMARK:** We call $d^c := -IdI$ **the twisted differential** and $dd^c$ **the pluri-Laplacian**. Hermitian metric is called **Gauduchon** if $dd^c(\omega^{n-1})=0$.

**THEOREM:** (Gauduchon) For each $(M,I,g)$, **there exists a unique**, up to a constant multiplier, **Gauduchon metric in the same conformal class.**

**REMARK:** This is very useful, because allows to define **a degree** of a holomorphic bundle, define stability, and prove a **non-Kähler version of Donaldson-Uhlenbeck-Yau therem.**

## Strongly Gauduchon metrics.

Let $(M, I, g)$ be a complex Hermitian $n$-manifold, $\omega \in \Lambda^{1,1}(M)$ its Hermitian form.

**DEFINITION:** Hermitian metric is called **strongly Gauduchon** if $\omega^{n-1}$ is a $(n-1, n-1)$-part of a closed form: $\omega^{n-1} = \eta^{n-1,n-1}$ (Dan Popovici).

**REMARK:** $dd^c$ preserves the Hodge type, hence $dd^c\eta^{p,q} = 0$ for any Hodge component $\eta^{p,q}$ of a closed form. **Therefore, strongly Gauduchon implies Gauduchon:** $dd^c(\eta^{n-1,n-1}) = -(d^c d\eta)^{n,n} = 0$ whenever $\eta$ is closed.

**Hermitian symplectic and SKT metrics.**

**DEFINITION:** A metric $g$ is called **SKT ("strong Kähler torsion")** or **pluriclosed** if $dd^c\omega = 0$.

**REMARK:** Such structures are used in physics (and differential geometry "generalized Kähler manifolds").

**DEFINITION:** A form $\omega$ is called **taming** or **symplectic-Hermitian** if it is a (1,1)-part of a symplectic form.

**REMARK:** **Symplectic-Hermitian implies pluriclosed,** by the same argument as used to show that strongly Gauduchon implies Gauduchon.

**REMARK:** Such metrics are used a lot in symplectic geometry. Also, Streets-Tian (2010) constructed a Ricci flow for symplectic-Hermitian metrics.

## Main results of today's talk

1. For an almost complex structure $I$ equipped with a taming symplectic form, all components of the space of complex curves are compact (Gromov). I will show that **the same is true for pluriclosed metrics, if $I$ is integrable.**

2. When $M$ has a quasi-line and a pluriclosed metric, it is actually **Moishezon.**

3. This is used to show that **a twistor space admits a pluriclosed metric if and only if it is tamed.**

4. I prove that no twistor space can admit a taming form, if it is non-Kähler. This implies that **twistor spaces are never pluriclosed.**

## Pluriharmonic function on the space of complex curves

**DEFINITION:** Let $S$ be a complex curve on a Hermitian manifold $(M, I, g, \omega)$. Define **the Riemannian volume** as $\operatorname{Vol}(S) := \int_S \omega$.

**DEFINITION:** A function $\varphi$ is called **pluriharmonic** if $dd^c\varphi = 0$.

**CLAIM:** Let $X$ be a component of the moduli of complex curves on a given complex manifold, $\tilde X$ the set of pairs $\{S \in X, z \in S \subset M\}$, ("the universal family"), and $\pi_M \tilde X \longrightarrow M$, $\pi_X \tilde X \longrightarrow X$ the forgetful maps. **Then the volume function $\operatorname{Vol}: X \longrightarrow \mathbb{R}^{>0}$ can be expressed as $\operatorname{Vol} = (\pi_X)_* \pi_M^* \omega$.**

**REMARK:** Since pullback and pushforward of differential forms commute with $d$, $d^c$, this gives $dd^c \operatorname{Vol} = (\pi_X)_* \pi_M^*(dd^c\omega)$. **Therefore, the volume function is pluriharmonic on $X$, whenever $\omega$ is pluriclosed.**

## Gromov's theorem

**THEOREM:** (Gromov) Let $M$ be a compact Hermitian almost complex manifold, $\mathfrak{X}$ the space of all complex curves on $M$, and $\mathfrak{X} \xrightarrow{\mathsf{Vol}} \mathbb{R}^{>0}$ the volume function. Then Vol is **proper** (preimage of a compact set is compact).

**COROLLARY:** Let $M$ be a complex manifold, equipped with a pluriclosed Hermitian form $\omega$, and $X$ a component of the moduli of complex curves. **Then the function $\mathsf{Vol}: X \longrightarrow \mathbb{R}^{>0}$ is constant, and $X$ is compact.**

**Proof:** Since $\mathsf{Vol} \geqslant 0$, the set $\mathsf{Vol}^{-1}(]-\infty, C])$ is compact for all $C \in \mathbb{R}$, hence Vol has a minimum somewhere in $X$. However, **a pluriharmonic function which has a minimum is necessarily constant** (E. Hopf's strong maximum principle). Therefore, Vol is constant: $\mathsf{Vol} = A$. Now, compactness of $X = \mathsf{Vol}^{-1}(A)$ follows from Gromov's theorem. ∎

## Complex manifolds and quasi-lines

**REMARK:** Let $S \subset M$ be a quasi-line. Then, for an appropriate tubular neighbourhood $U \subset M$ of $S$, **"for every two points $x, y \in U$ close to $S$ and far from each other, there is a unique deformation of $S$ containing $X$ and $Y$."**

More precisely:

**Claim 1:** Let $S \subset M$ be a quasi-line. Then, for an appropriate tubular neighbourhood $U \subset M$ of $S$, and any open neighbourhood $W$ of a diagonal $\Delta \subset U \times U$, there exists a smaller open neighbourhood $V$, of $S$ inside $U$ such that for each $(x, y) \in V \times V \backslash W$, **there exists a unique deformation $S' \subset U$ of $S$ containing $x$ and $y$.**

Similarly,

**Claim 2:** A small deformation $S' \subset U$ of $S$ passing through $z \in S$ **is uniquely determined by a 1-jet of $S'$ at $z$.**

## Quasilines in Moishezon manifolds

**THEOREM:** Let $M$ be a complex manifold, $S \subset M$ a quasi-line, and $W$ its deformation space. Assume that $W$ is compact. **Then $M$ is Moishezon.**

**Proof. Step 1:** Let $z \in M$ a point, containing a quasi-line $S \in W$, $W_z$ the set of all curves $S_1 \in W$ containing $z$, and $\tilde{W}_z$ – the set of all pairs $\{x \in S_1, S_1 \in W_z\}$. From Claim 1, **it follows that the map $\tilde{W}_z \longrightarrow M$, $(S, x) \longrightarrow x$ is surjective and finite at generic point.**

**Step 2:** Therefore, **it would suffice to prove that $\tilde{W}_z$ is Moishezon.**

**Step 3:** After an appropriate bimeromorphic transform, we may assume that $\tilde{W}_z \longrightarrow W_z$ is a smooth, proper map with rational, 1-dimensional fibers. **Then $\tilde{W}_z$ is Moishezon $\Leftrightarrow$ $W_z$ is Moishezon.**

**Step 4:** By Claim 2, the map from $W_z$ to $\mathbb{P}T_zM$ mapping a quasi-line to its 1-jet is also generically finite. **Therefore, $W_z$ is Moishezon.** ■

**COROLLARY:** Let $M$ be a twistor space admitting a pluriclosed Hermitian metric. **Then $M$ is Moishezon.** ■

## Harvey-Lawson duality argument applied to pluriclosed metrics

Recall the classical theorem

**THEOREM:** (Harvey-Lawson, 1982) Let $M$ be a compact, complex manifold. Then the **following conditions are equivalent.**

a. $M$ **does not admit a Kähler metric.**

b. $M$ **has a non-zero, positive** $(n-1, n-1)$**-current** $\Theta$ **which is a** $(n-1, n-1)$**-part of a closed current.**

The same argument, applied to pluriclosed or taming metrics, brings the following.

**THEOREM 1:** Let $M$ be a compact, complex manifold. Then

1. $M$ **admits no pluriclosed metrics** $\Leftrightarrow$ $M$ **admits a positive,** $dd^c$**-exact** $(n-1, n-1)$**-current**

2. $M$ **admits no Hermitian symplectic metrics** $\Leftrightarrow$ $M$ **admits a positive, exact** $(n-1, n-1)$**-current.** ■

## Applications.

**COROLLARY:** Any twistor space which admits a pluriclosed metric, **also admits a Hermitian symplectic structure.**

**Proof:** Since $M$ is Moishezon, it satisfies $dd^c$-lemma (Deligne-Griffiths-Morgan-Sullivan). Therefore, **any positive, exact $(n-1, n-1)$-current is $dd^c$-exact.** ■

**COROLLARY:**
**No non-Kähler twistor space admits a pluriclosed (or Hermitian symplectic) metric.**

**Proof:** Th. Peternell (1984) has shown that any non-Kähler Moishezon $n$-manifold admits an exact, positive $(n-1, n-1)$-current. Therefore, it is never Hermitian symplectic (Theorem 1). ■