Rational curves on non-Kähler manifolds

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Plan

1. Non-Kähler manifolds. Twistor spaces.

2. Different Hermitian structures and conditions.

3. Rational curves on twistor spaces and special metrics.

4. Applications.

Non-Kähler manifolds

Constructions of compact, non-Kähler manifolds.

1. **Locally conformally Kähler manifolds.** A quotient of a Kähler manifold by a discrete group acting by homotheties. **Never admits a Kähler metric.** (Vaisman).

Example: Hopf manifold $(\mathbb{C}^n \backslash 0)/\mathbb{Z}$.

2. **Left-invariant complex structures** on Lie groups or their quotients by discrete groups. **Almost never Kähler** (exception: a torus).

3. **Twistor spaces**. Huge. **Any finite-generated group can be a fundamental of a twistor space** (Taubes; Panov-Petrunin). **Never Kähler** (Hitchin), except $\mathbb{C}P^3$ and flag spaces.

4. Moishezon (and Fujiki class C) manifolds.

Twistor spaces (hyperkähler geometry)

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata)

EXAMPLE: If $M = \mathbb{H}^n$, $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$ (total space of a vector bundle $(\mathcal{O}(1)^{\oplus n})$.

REMARK: For M compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.

Twistor spaces (4-manifolds)

DEFINITION: Let M be a Riemannian 4-manifold. Consider the action of the Hodge $*$-operator: $* : \Lambda^2 M \longrightarrow \Lambda^2 M$. Since $*^2 = 1$, the eigenvalues are ± 1, and one has a decomposition $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$ onto **autodual** $(*\eta = \eta)$ and **anti-autodual** $(*\eta = -\eta)$ forms.

REMARK: If one changes the orientation of M, leaving metric the same, $\Lambda^+ M$ and $\Lambda^- M$ are exchanged. **Therefore,** $\dim \Lambda^2 M = 6$ **implies** $\dim \Lambda^\pm(M) = 3$**.**

REMARK: Using the isomorphism $\Lambda^2 M = \mathfrak{so}(TM)$, we interpret $\eta \in \Lambda^2_m M$ as an endomorphisms of $T_m M$. **Then the unit vectors $\eta \in \Lambda^+_m M$ correspond to oriented, orthogonal complex structures on $T_m M$.**

DEFINITION: Let $\operatorname{Tw}(M) := S\Lambda^+ M$ be the set of unit vectors in $\Lambda^+ M$. At each point $(m, s) \in \operatorname{Tw}(M)$, consider the decoposition $T_{m,s} \operatorname{Tw}(M) = T_m M \oplus T_s S\Lambda^+_m M$, induced by the Levi-Civita connection. Let I_s be the complex structure on $T_m M$ induced by s, $I_{S\Lambda^+_m M}$ the complex structure on $S\Lambda^+_m M = S^2$ induced by the metrics and orientation, and $\mathcal{I} : T_{m,s} \operatorname{Tw}(M) \longrightarrow T_{m,s} \operatorname{Tw}(M)$ be equal to $\mathcal{I}_s \oplus I_{S\Lambda^+_m M}$. An almost complex manifold $(\operatorname{Tw}(M), \mathcal{I})$ is called **the twistor space** of M.

PROPERTIES OF TWISTOR SPACES:

1. The almost complex structure on Tw(M),$\mathcal{I}$ **is a conformal invariant** of M. Moreover, **one can reconstruct the conformal structure** on M from the almost complex structure on Tw(M) and its anticomplex involution $(m, s) \longrightarrow (m, -s)$.

2. Tw(M),$\mathcal{I}$ **is a complex manifold if and only if $W^+ = 0$,** where W^+ ("self-dual conformal curvature") is an autodual component of the curvature tensor. Such manifolds are called **conformally half-flat** or **ASD (anti-selfdual).**

3. For a hyperkähler 4-fold, two definitions of twistor spaces coincide.

Rational curves on $\operatorname{Tw}(M)$.

DEFINITION: An ample rational curve on a complex manifold M is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$, with $i_k > 0$. It is called **a quasi-line** if all $i_k = 1$.

CLAIM: Let M be a compact complex manifold containing a an ample rational line. **Then any N points $z_1, ..., z_N$ can be connected by an ample rational curve.**

CLAIM: Let M be a Riemannian 4-manifold, $\operatorname{Tw}(M) \xrightarrow{\sigma} M$ its twistor space, $m \in M$ a point, and $S_m := \sigma^{-1}(m) = S\Lambda^+_m(M)$ the corresponding S^2 in $\operatorname{Tw}(M)$. **Then S_m is a quasi-line.**

Proof: Since the claim is essentially infinitesimal, it suffices to check it when M is flat. **Then $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus 2}) \cong \mathbb{C}P^3 \backslash \mathbb{C}P^1$, and S_m is a section of $\mathcal{O}(1)^{\oplus 2}$.** ■

Hermitian structures on complex manifolds.

Let (M, I, g) be a complex Hermitian n-manifold, $\omega \in \Lambda^{1,1}(M)$ its Hermitian form.

REMARK: For each $1 \leqslant k \leqslant n-1$, the condition $d(\omega^k) = 0$ implies $d\omega = 0$. Hermitian metric is called **balanced** if $d(\omega^{n-1}) = 0$. All twistor spaces are balanced (Hitchin). All Moishezon manifolds are balanced (Alessandrini-Bassaneli).

REMARK: We call $d^c := -IdI$ **the twisted differential** and dd^c **the pluri-Laplacian**. Hermitian metric is called **Gauduchon** if $dd^c(\omega^{n-1}) = 0$.

THEOREM: (Gauduchon) For each (M, I, g), **there exists a unique**, up to a constant multiplier, **Gauduchon metric in the same conformal class.**

REMARK: This is very useful, because allows to define **a degree** of a holomorphic bundle, define stability, and prove a **non-Kähler version of Donaldson-Uhlenbeck-Yau therem.**

Strongly Gauduchon metrics.

Let (M, I, g) be a complex Hermitian n-manifold, $\omega \in \Lambda^{1,1}(M)$ its Hermitian form.

DEFINITION: Hermitian metric is called **strongly Gauduchon** if ω^{n-1} is a $(n-1, n-1)$-part of a closed form: $\omega^{n-1} = \eta^{n-1,n-1}$ (Dan Popovici).

REMARK: dd^c preserves the Hodge type, hence $dd^c\eta^{p,q} = 0$ for any Hodge component $\eta^{p,q}$ of a closed form. **Therefore, strongly Gauduchon implies Gauduchon:** $dd^c(\eta^{n-1,n-1}) = -(d^c d\eta)^{n,n} = 0$ whenever η is closed.

Hermitian symplectic and SKT metrics.

DEFINITION: A metric g is called **SKT ("strong Kähler torsion")** or **pluriclosed** if $dd^c\omega = 0$.

REMARK: Such structures are used in physics (and differential geometry "generalized Kähler manifolds").

DEFINITION: A form ω is called **taming** or **symplectic-Hermitian** if it is a (1,1)-part of a symplectic form.

REMARK: **Symplectic-Hermitian implies pluriclosed,** by the same argument as used to show that strongly Gauduchon implies Gauduchon.

REMARK: Such metrics are used a lot in symplectic geometry. Also, Streets-Tian (2010) constructed a Ricci flow for symplectic-Hermitian metrics.

Main results of today's talk

1. For an almost complex structure I equipped with a taming symplectic form, all components of the space of complex curves are compact (Gromov). I will show that **the same is true for pluriclosed metrics, if I is integrable.**

2. When M has a quasi-line and a pluriclosed metric, it is actually **Moishezon.**

3. This is used to show that **a twistor space admits a pluriclosed metric if and only if it is tamed.**

4. I prove that no twistor space can admit a taming form, if it is non-Kähler. This implies that **twistor spaces are never pluriclosed.**

Pluriharmonic function on the space of complex curves

DEFINITION: Let S be a complex curve on a Hermitian manifold (M, I, g, ω). Define **the Riemannian volume** as $\operatorname{Vol}(S) := \int_S \omega$.

DEFINITION: A function φ is called **pluriharmonic** if $dd^c\varphi = 0$.

CLAIM: Let X be a component of the moduli of complex curves on a given complex manifold, $\tilde X$ the set of pairs $\{S \in X, z \in S \subset M\}$, ("the universal family"), and $\pi_M\ \tilde X \longrightarrow M$, $\pi_X\ \tilde X \longrightarrow X$ the forgetful maps. **Then the volume function $\operatorname{Vol}: X \longrightarrow \mathbb{R}^{>0}$ can be expressed as $\operatorname{Vol} = (\pi_X)_* \pi_M^* \omega$.**

REMARK: Since pullback and pushforward of differential forms commute with d, d^c, this gives $dd^c \operatorname{Vol} = (\pi_X)_* \pi_M^*(dd^c\omega)$. **Therefore, the volume function is pluriharmonic on X, whenever ω is pluriclosed**.

Gromov's theorem

THEOREM: (Gromov) Let M be a compact Hermitian almost complex manifold, $\mathfrak{X}$ the space of all complex curves on M, and $\mathfrak{X} \xrightarrow{\text{Vol}} \mathbb{R}^{>0}$ the volume function. Then Vol is **proper** (preimage of a compact set is compact).

COROLLARY: Let M be a complex manifold, equipped with a pluriclosed Hermitian form ω, and X a component of the moduli of complex curves. **Then the function $\text{Vol}: X \longrightarrow \mathbb{R}^{>0}$ is constant, and X is compact.**

Proof: Since $\text{Vol} \geqslant 0$, the set $\text{Vol}^{-1}(]-\infty, C])$ is compact for all $C \in \mathbb{R}$, hence Vol has a minimum somewhere in X. However, **a pluriharmonic function which has a minimum is necessarily constant** (E. Hopf's strong maximum principle). Therefore, Vol is constant: $\text{Vol} = A$. Now, compactness of $X = \text{Vol}^{-1}(A)$ follows from Gromov's theorem. ∎

Complex manifolds and quasi-lines

REMARK: Let $S \subset M$ be a quasi-line. Then, for an appropriate tubular neighbourhood $U \subset M$ of S, **"for every two points $x, y \in U$ close to S and far from each other, there is a unique deformation of S containing X and Y."**

More precisely:

Claim 1: Let $S \subset M$ be a quasi-line. Then, for an appropriate tubular neighbourhood $U \subset M$ of S, and any open neighbourhood W of a diagonal $\Delta \subset U \times U$, there exists a smaller open neighbourhood V, of S inside U such that for each $(x, y) \in V \times V \backslash W$, **there exists a unique deformation $S' \subset U$ of S containing x and y.**

Similarly,

Claim 2: A small deformation $S' \subset U$ of S passing through $z \in S$ **is uniquely determined by a 1-jet of S' at z.**

Quasilines in Moishezon manifolds

THEOREM: Let M be a complex manifold, $S \subset M$ a quasi-line, and W its deformation space. Assume that W is compact. **Then M is Moishezon.**

Proof. Step 1: Let $z \in M$ a point, containing a quasi-line $S \in W$, W_z the set of all curves $S_1 \in W$ containing z, and $\tilde{W}_z$ – the set of all pairs $\{x \in S_1, S_1 \in W_z\}$. From Claim 1, **it follows that the map $\tilde{W}_z \longrightarrow M$, $(S, x) \longrightarrow x$ is surjective and finite at generic point.**

Step 2: Therefore, **it would suffice to prove that $\tilde{W}_z$ is Moishezon.**

Step 3: After an appropriate bimeromorphic transform, we may assume that $\tilde{W}_z \longrightarrow W_z$ is a smooth, proper map with rational, 1-dimensional fibers. **Then $\tilde{W}_z$ is Moishezon $\Leftrightarrow$ W_z is Moishezon.**

Step 4: By Claim 2, the map from W_z to $\mathbb{P}T_zM$ mapping a quasi-line to its 1-jet is also generically finite. **Therefore, W_z is Moishezon.** ∎

COROLLARY: Let M be a twistor space admitting a pluriclosed Hermitian metric. **Then M is Moishezon.** ∎

Harvey-Lawson duality argument applied to pluriclosed metrics

Recall the classical theorem

THEOREM: (Harvey-Lawson, 1982) Let M be a compact, complex manifold. Then the **following conditions are equivalent.**

a. M **does not admit a Kähler metric.**

b. M **has a non-zero, positive** $(n-1,n-1)$**-current** Θ **which is a** $(n-1,n-1)$**-part of a closed current.**

The same argument, applied to pluriclosed or taming metrics, brings the following.

THEOREM 1: Let M be a compact, complex manifold. Then

1. M **admits no pluriclosed metrics** $\Leftrightarrow$ M **admits a positive,** dd^c**-exact** $(n-1,n-1)$**-current**

2. M **admits no Hermitian symplectic metrics** $\Leftrightarrow$ M **admits a positive, exact** $(n-1,n-1)$**-current.** ■

Applications.

COROLLARY: Any twistor space which admits a pluriclosed metric, **also admits a Hermitian symplectic structure.**

Proof: Since M is Moishezon, it satisfies dd^c-lemma (Deligne-Griffiths-Morgan-Sullivan). Therefore, **any positive, exact $(n-1,n-1)$-current is dd^c-exact.** ■

COROLLARY:
No non-Kähler twistor space admits a pluriclosed (or Hermitian symplectic) metric.

Proof: Th. Peternell (1984) has shown that any non-Kähler Moishezon n-manifold admits an exact, positive $(n-1,n-1)$-current. Therefore, it is never Hermitian symplectic (Theorem 1). ■