

Barge-Ghys quasimorphisms taking values in semi-direct products

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Abstract. A (non-commutative) Ulam quasimorphism is a map q from a group Γ to a topological group G such that $q(xy)q(y)^{-1}q(x)^{-1}$ belongs to a fixed compact subset of G . Generalizing the construction of Barge and Ghys, we build a family of quasimorphisms on a fundamental group of a closed manifold M of negative sectional curvature, taking values in G , which is a semi-direct product of a non-compact Lie group A and a compact Lie group K . This construction, which generalizes the Barge-Ghys quasimorphisms, associates a quasimorphism to any principal G -bundle with connection whose curvature is a 2-form with coefficients in the Lie algebra of K .

Kapovich and Fujiwara have shown that all quasimorphisms taking values in a discrete group can be constructed from group homomorphisms and quasimorphisms taking values in a commutative group. We construct Barge-Ghys type quasimorphisms taking prescribed values on a given subset in Γ . When A is abelian, this gives counterexamples to the Kapovich and Fujiwara theorem for quasimorphisms taking values in a Lie group. Our construction also generalizes a result proven by D. Kazhdan in his paper “On ε -representations”. Kazhdan has proved that for any $\varepsilon > 0$, there exists an ε -representation of the fundamental group of a Riemann surface of genus 2 which cannot be $1/10$ -approximated by a representation. We generalize his result by constructing an ε -representation of the fundamental group of a closed manifold of negative sectional curvature taking values in G . This paper serves as an erratum to “Non-commutative Barge-Ghys quasimorphisms” where these results were erroneously claimed for nilpotent groups.

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1 Introduction

1.1 The main results

We start by citing the main results of this paper. We introduce the relevant definitions and motivation for these notions later, and here we just give the statements. The following theorem complements the main result of [FK] by finding a counterexample to their theorem when the quasimorphism takes values in a Lie group.

Theorem 4.14: Let $G := A \rtimes K$ be a semidirect product of a compact group K and an abelian Lie group $A = \mathbb{R}^n, n \geq 2$, and K acts transitively on a sphere in A . Let $\Gamma := \pi_1(M)$, where M is a closed manifold of strictly negative sectional curvature, $\dim_{\mathbb{R}} M > 1$. Then there exists a non-constructible HBG-quasimorphism $q_{\nabla} : \Gamma \rightarrow G$.

The HBG-quasimorphisms are defined in [Definition 3.25](#) (see also [Remark 3.26](#)), and constructibility in [Definition 4.12](#), following [\[FK\]](#). Note that we define the HBG-quasimorphisms in more relaxed assumptions, [\(1.2\)](#), where A is not necessarily abelian. However, to prove the non-constructibility in a more economical way, we require A to be abelian.

Another theorem generalizes one of the results of Kazhdan, [\[Kaz\]](#).

Theorem 5.3: Let M be a closed manifold of strictly negative sectional curvature, G a positive-dimensional compact Lie group, and P a trivial principal G -bundle. For any connection ∇ in P , let $q_\nabla : \pi_1(M) \rightarrow G$ denote the corresponding HBG-quasimorphism ([Definition 3.25](#)). Choose a left-invariant metric on G such that the diameter of any closed subgroup is at least $1/3$. Then for each $\varepsilon > 0$, there exists a connection ∇ such that q_∇ is an ε -representation which cannot be $1/3$ -approximated by a representation.

We explain the terms “ ε -representation” and “ δ -approximation” in [Subsection 1.6](#), following [\[Kaz\]](#).

1.2 Ulam quasimorphisms

Throughout this paper, $G = A \rtimes K$ is a Lie group, obtained as a semidirect product of a non-compact Lie group A (often, but not always, A is assumed to be abelian) and a compact Lie group K acting on A . Usually we assume that G is connected, though this assumption is not always necessary.

Definition 1.1: Let G be a Lie group, and Γ any group. **An Ulam quasimorphism** $q : \Gamma \rightarrow G$ is a map which satisfies $q(x^{-1}) = q(x)^{-1}$ and $q(x)q(y) \in C \cdot q(xy)$, where $C \subset G$ is a fixed compact, independent from the choice of x, y . Two Ulam quasimorphisms $q_1, q_2 : \Gamma \rightarrow G$ are called **equivalent** if there exists a compact subset $C \subset G$ such that $q_1(x) \in C \cdot q_2(x)$, for all $x \in \Gamma$.

In this paper, we give a versatile construction of Ulam quasimorphisms associated with connections in vector bundles over a closed manifold M of strictly negative sectional curvature. As a result, we obtain a quasimorphism from $\pi_1(M)$ to a Lie group $G := A \rtimes K$, called a *Barge-Ghys quasimorphism*. This allows us to find quasimorphisms which are not constructible, in the sense of Fujiwara and Kapovich [\[FK\]](#). The Fujiwara-Kapovich constructible quasimorphisms are obtained from quasimorphisms taking values in abelian

groups. Unlike the constructible quasimorphisms, the Barge-Ghys quasimorphisms we construct cannot be obtained from ones taking values in abelian groups ([Theorem 4.14](#)).

Interestingly enough, Fujiwara-Kapovich have proved that *any* Ulam quasimorphism taking values in a discrete group is constructible. It turns out that the Ulam quasimorphisms taking values in Lie groups are of entirely different nature.

Ulam quasimorphisms from a free group to a Lie group G with bi-invariant metric were explored by P. Rolli in [\[R\]](#). Rolli has constructed non-trivial quasimorphisms from a free group to G . It seems that Rolli's construction also can lead to non-constructible quasimorphisms.

As an application of our approach, we give a construction of ε -representations which cannot be approximated by a representation ([Theorem 5.3](#)), generalizing a result of Kazhdan ([\[Kaz\]](#)).

Note that our definition of Ulam quasimorphism may not be the optimal for some purposes. Fujiwara and Kapovich [\[FK\]](#) give several non-equivalent (more relaxed) definitions of a quasimorphism taking values in a non-abelian group G ; for abelian G , all these definitions are equivalent. Another, even more relaxed, definition was considered in [\[HS\]](#).

The most recent contribution to this subject is due to S. Douba, F. Fournier-Facio, S. Hughes, S. Machado ([\[DFHM\]](#)), who responded to our previous publication [\[BV\]](#). In addition to correcting erroneous claims in [\[BV\]](#), they proved a number of powerful statements constraining the behaviour of non-commutative Ulam quasimorphisms. They prove that any Ulam quasimorphism $q : \Gamma \rightarrow G$ to an algebraic group G which has no compact algebraic subquotients is constructible ([\[DFHM, Theorem E\]](#)).

This result rules out the existence of non-constructible quasimorphisms $q : \Gamma \rightarrow G$ for simply connected nilpotent Lie groups G , which were claimed to exist in [\[BV\]](#).

However, in the same paper ([\[DFHM, Proposition 6.1\]](#)), Douba, Fournier-Facio, Hughes and Machado construct a non-constructible quasimorphism from an acylindrically hyperbolic group which surjects onto $\mathbb{Z}$ to $\mathbb{R}^2 \rtimes SO(2)$. This example relies on Corollary 1.2 from [\[BBF\]](#), which uses a construction similar to Brooks' ([\[Br\]](#)). We do not know whether the Barge-Ghys quasimorphisms can be used to emulate this example.

1.3 Quasimorphisms, bounded cohomology and the commutator length

In the literature, the quasimorphisms are usually considered as maps taking values in $\mathbb{R}$. In this context, a quasimorphism is defined as a map $q : G \rightarrow \mathbb{R}$ which satisfies $|q(xy) - q(x) - q(y)| < C$, where C is a constant independent from x, y . We shall sometimes call such maps **commutative quasimorphisms**.

In geometric context, this notion originates in the paper of Gromov [G2]. Using the quasimorphisms, R. Brooks proved that the second bounded cohomology $H_b^2(\mathbb{F}_2, \mathbb{R})$ of the free group $\mathbb{F}_2$ is infinite-dimensional ([Br, G2]). Since then, quasimorphisms $q : G \rightarrow \mathbb{R}$ became prevalent in topology ([C]), symplectic geometry ([EP, Sh]) and dynamics ([BM, GG]).

Barge and Ghys ([BG]) generalized the observation of Brooks to prove that the second bounded cohomology of $\pi_1(S)$ is infinite-dimensional for any Riemann surface S with $g(S) > 1$.

We say that two quasimorphisms $q, q_1 : G \rightarrow \mathbb{R}$ are **equivalent** if $|q(x) - q_1(x)| < C$, where C is a constant independent from x . The group $\mathcal{Q}$ of equivalence classes of quasimorphisms fits into an exact sequence

$$H^1(G, \mathbb{R}) \rightarrow \mathcal{Q} \rightarrow H_b^2(G, \mathbb{R}) \rightarrow H^2(G, \mathbb{R}), \quad (1.1)$$

where $H_b^2(G, \mathbb{R})$ denotes the bounded cohomology. A quasimorphism $q : G \rightarrow \mathbb{R}$ is called **homogeneous** if it satisfies $q(x^n) = nq(x)$ for any $x \in G$ and $n \in \mathbb{Z}$. It is possible to see that every quasimorphism $q : G \rightarrow \mathbb{R}$ is equivalent to a unique homogeneous quasimorphism ([C, Lemma 2.2.1]).

The commutator length of $g \in [G, G]$ is the minimal number m such that g can be represented as a product of m commutators. Recall that **the stable commutator length** of an element $g \in G$ is defined as

$$\text{scl}(g) := \lim_{n \rightarrow \infty} \frac{\text{cl}(g^n)}{n},$$

where cl is the commutator length. The homogeneous quasimorphisms are related with the stable commutator length, due to the celebrated theorem of Bavard ([B]). This result is known as *Bavard duality*. In its simplest version, Bavard duality can be stated as follows: for any $g \in [G, G]$, g has non-zero stable commutator length if and only if there exists a homogeneous quasimorphism $q : G \rightarrow \mathbb{R}$ such that $q(g) \neq 0$.

1.4 Barge-Ghys construction and manifolds of strictly negative curvature

Further on, we are interested in vector bundles over closed manifolds of strictly negative sectional curvature. Sometimes such manifolds are called “hyperbolic”, but we don’t use this term to avoid confusion with manifolds of constant strictly negative curvature.

We study the Ulam quasimorphisms associated with the holonomy of a connection on vector bundles or on principal G -bundles, satisfying the following properties.

First, $G = A \rtimes K$ is a connected Lie group, obtained as a semidirect product of a non-compact Lie group A with a compact group K . Second, the curvature of the connection is a 2-form with coefficients in the Lie algebra of K . (1.2)

From now on, we tacitly assume that the Lie group G is as above, $G = A \rtimes K$.

In the sequel, we use either the language of G -bundles or the language of vector bundles with connection, whatever is more convenient. However, all statements that we make can be easily translated from one language to another, giving equivalent results in two parallel frameworks.

Remark 1.2: We always assume that the base manifold of strictly negative curvature has dimension at least 2.

Barge-Ghys quasimorphisms were introduced in [BG], who used them to prove that the bounded cohomology of a Riemann surface is infinite-dimensional. In the literature these quasimorphisms are variously called de Rham quasimorphisms ([C]) or Barge-Ghys quasimorphisms [EP, PR]; we follow the second convention.

The generalization of Barge-Ghys quasimorphisms to closed manifolds of strictly negative curvature seems to be well known ([Mar]). However, the infinite-dimensionality of the space of Barge-Ghys quasimorphisms, originally proven in [BG] for Riemann surface, is less straightforward. This result was established in [BFMSS].

1.5 Barge-Ghys quasimorphisms associated with a connection

We feel compelled to make a few comments about the terminology used in this paper.

Originally, the Barge-Ghys quasimorphisms were associated with a non-closed 1-form θ on a closed manifold M of strictly negative sectional curvature. Given $\gamma \in \pi_1(M, x)$, we represent γ by a geodesic loop $\underline{\gamma}$ (which is unique, because M has negative curvature), and put $q(\gamma) = \int_{\underline{\gamma}} \theta$.

Taking θ as a connection form in a trivial line bundle, the number $\int_{\underline{\gamma}} \theta$ is interpreted as the holonomy of this connection along $\underline{\gamma}$ (Subsection 3.2).

In Subsection 3.2, we generalize this construction to principal G -bundles with connection satisfying (1.2) and their associate vector bundles. This gives a “non-commutative quasimorphism” (Definition 1.1), which we call “a non-commutative Barge-Ghys quasimorphism associated with a connection”.

This quasimorphism does not satisfy $q(x^n) = q(x)^n$, that is, it is not homogeneous (Definition 3.23). In the usual theory of quasimorphisms (which we call throughout this paper “commutative quasimorphisms”) for every quasimorphism there exists a unique homogeneous quasimorphism in the same equivalence class. This construction is called “homogenization”; it is obtained by the standard limit construction ([C, Lemma 2.2.1]).

Unfortunately, we were unable to generalize the homogenization construction to the non-commutative case. Instead we use an ad hoc construction, which works in the same generality, and gives a homogeneous quasimorphism (Subsection 3.5). The quasimorphisms obtained this way are called “the homogeneous Barge-Ghys quasimorphisms”. They are also associated with a connection in a vector bundle or in a principal G -bundle.

There are disclaimers we need to make at this point. First of all, we could not devise a general definition of “Barge-Ghys quasimorphisms”. These two kinds of Barge-Ghys quasimorphisms are the only cases we could come up with.

Second, the Barge-Ghys quasimorphisms associated with a connection are distinct from the homogeneous Barge-Ghys quasimorphisms. These are two distinct classes which rarely intersect, and we do not know if they can be united in a meaningful class of more general quasimorphisms. However, each Barge-Ghys quasimorphism associated with a G -bundle with connection is equivalent to the homogeneous Barge-Ghys quasimorphism associated with the same connection (Claim 3.30).

1.6 Ulam stability and Kazhdan’s ε -representations

The earliest mention of quasimorphisms is found in Ulam’s 1960 book “A collection of mathematical problems”, Chapter 6 ([U]). Ulam defined an

ε -automorphism of a topological group as a map $\rho : G \longrightarrow G$ such that $\rho(xy)\rho(x)^{-1}\rho(y)^{-1}$ belongs to an ε -neighbourhood of the identity. Ulam asked whether any such map admits an $k\varepsilon$ -approximation by an automorphism, for some $k > 0$ which is independent from ε .

More recently, this problem was generalized to representations (see [Definition 1.3](#) below). However, the notion of Ulam stability seems to be present, implicitly, in earlier works of von Neumann ([\[vN\]](#)) and Turing ([\[Tu\]](#)).

A similar question was considered in 1982 by D. Kazhdan ([\[Kaz\]](#)). He defined an ε -**representation** of a group G as a map $\rho : G \longrightarrow U(V)$, where V is a Hilbert space, finite dimensional or infinite dimensional, satisfying

$$\|\rho(xy) - \rho(x)\rho(y)\| < \varepsilon,$$

where $\|\cdot\|$ is the operator norm. The distance between two maps $\rho_1, \rho_2 : G \longrightarrow U(V)$ can be defined as

$$d(\rho_1, \rho_2) := \sup_{x \in G} \|\rho_1(x) - \rho_2(x)\|.$$

Following a suggestion of V. Milman, Kazhdan asked whether for any $\delta > 0$ there exists $\varepsilon > 0$ such that any ε -representation can be δ -approximated by a representation. He proved that this holds true for amenable groups. When $G = \pi_1(S)$, where S is a genus 2 Riemann surface, Kazhdan has constructed an ε -representation $G \longrightarrow U(n)$, which cannot be $1/10$ -approximated by a representation, for any given $\varepsilon > 0$.

Definition 1.3: The group Γ is called **Ulam stable** ([\[BOT\]](#)) if for any $\delta > 0$ there exists $\varepsilon > 0$ such that any finite-dimensional ε -representation $q : \Gamma \longrightarrow U(V)$ can be δ -approximated by a representation $\rho : \Gamma \longrightarrow U(V)$. It is called **strongly Ulam stable** if the same is true even for infinite-dimensional Hilbert representations.

In [\[Kaz\]](#), D. Kazhdan has proven that all amenable groups are strongly Ulam stable. Using the Barge-Ghys quasimorphisms, we were able to prove this result for the fundamental group of any closed strictly negatively curved manifold. In [Theorem 5.3](#), we generalize [\[Kaz, Theorem 2\]](#) showing that a fundamental group Γ of a complete Riemannian manifold with uniformly bounded strictly negative sectional curvature admits an ε -representation taking values in a compact Lie group G , which cannot be $1/3$ -approximated by a representation¹. Kazhdan proves this for $\Gamma = \pi_1(S)$, where S is a genus

¹Our norm conventions are different from Kazhdan's, but the constants $1/3$ and $1/10$ depend on the choice of normalizations, and are not important in themselves.

2 Riemann surface, and $G = U(n)$.

In [BOT], Burger, Ozawa and Thom address the question of strong Ulam stability, obtaining definite results for a large class of groups. Let Γ be a group which contains a subgroup Λ admitting a homogeneous $\mathbb{R}$ -valued quasimorphism which is not a homomorphism, that is, such that the map

$$H_b^2(\Lambda, \mathbb{R}) \longrightarrow H^2(\Lambda, \mathbb{R})$$

has non-trivial kernel. Then Γ has an infinite-dimensional Hilbert ε -representation, for any given $\varepsilon > 0$, which cannot be $\frac{\sqrt{3}}{16}$ -approximated by a representation. Also, they observe that any free group admits a finite-dimensional ε -representation taking values in $U(n)$ which cannot be 2-approximated by a representation ([BOT, Proposition 3.3]).

In [GLMR], the Ulam stability was cast in a cohomological setting. Glebsky, Lubotzky, Monod and Rangarajan defined an asymptotic version of bounded cohomology, and proved that vanishing of asymptotic cohomology implies Ulam stability. In the same paper, Ulam stability of $U(1)$ -representations is directly related to existence of quasimorphisms.

1.7 Constructible quasimorphisms

The paper [FK] by Fujiwara and Kapovich is the fundamental treatment of Ulam quasimorphisms taking values in a non-commutative group G . Fujiwara and Kapovich considered the case when G is discrete, and their result is mostly negative. Fujiwara and Kapovich defined “constructible quasimorphisms”, which are up to equivalence and finite index quasimorphisms which can be constructed in terms of homomorphisms, quasimorphisms with abelian target, and sections of bounded central extensions.

Then they proved that any Ulam quasimorphism taking values in a discrete group is always constructible ([FK, Theorem 1.2]). Results of Fujiwara and Kapovich were extended to quasimorphisms taking values in an algebraic Lie group without compact subquotients in [DFHM].

In Theorem 4.14 we construct examples of quasimorphisms taking values in $G := A \rtimes K$, which do not satisfy conclusions of [FK, Theorem 1.2] and [DFHM, Theorem E]. We prove that for any closed manifold M of strictly negative sectional curvature, there exists an Ulam quasimorphism $\pi_1(M) \longrightarrow G$ which is not constructible.

1.8 History of this paper

This paper supersedes [BV], which is found to contain subtle errors, forcing us to rewrite most statements and a few of the proofs. We are grateful to the authors of [DFHM], who have extensively developed the methods of [FK] and discovered that some of the results of [FK] are valid for quasimorphisms taking values in Lie groups. They notified us of the contradictions between their results and ours, encouraging us to review and eventually rewrite [BV]. Some of their arguments and results were already presented in Subsection 1.2.

Their results invalidate quite a few of results of [BV], where we dealt with simply connected nilpotent Lie groups which have no compact subquotients. After some reflection, we have discovered a way to correct [BV], adding an extra condition (1.2). Miraculously, the rest of [BV] survives, with the exception of the section on comparison theorems between different types of quasimorphism (Ulam, algebraic, geometric), which is removed from the current paper.

2 Principal bundles and vector bundles: basic facts and conventions

Almost everything we say in this paper can be formulated for vector bundles or for principal G -bundles, where G is a Lie group. Usually we state only one of two versions, leaving the rest for the reader. In this section, we briefly explain the passage from vector bundles with connection to G -bundles and vice versa.

Definition 2.1: Let G be a Lie group. A **principal G -bundle** is a smooth manifold E equipped with a smooth, free G -action, such that the natural map $E \rightarrow E/G$ is a locally trivial fibration.

Example 2.2: Consider the standard action of $U(1)$ on 3-dimensional sphere $S^3 \subset \mathbb{C}^2$, with e^{it} taking (ξ_1, ξ_2) to $(e^{it}\xi_1, e^{it}\xi_2)$. Then $S^3/U(1) = S^2$, and this action defines a principal $U(1)$ -bundle on a 2-dimensional sphere. This fibration is known as **Hopf fibration**.

Definition 2.3: Let $\pi : E \rightarrow M$ be a principal G -bundle and V a space with G -action. Consider the quotient $(E \times V)/G$, where G acts diagonally. Since the action of G is free on fibers of π , the quotient $(E \times V)/G$ is a locally

trivial fibration on M with fiber V . It is called **the associated fibration**.

Example 2.4: Let V be a representation of a group G , and $\pi : E \rightarrow M$ be a principal G -bundle. Then the associated fibration $(E \times V)/G$ is a vector bundle, called **a vector bundle with a G -structure**. We say that **the structure group of a vector bundle B is reduced to G** if B is obtained from a principal G -bundle this way.

Example 2.5: Consider a complex vector bundle B over M equipped with a Hermitian structure, and let $\pi : E \rightarrow M$ be the space of all orthonormal complex frames in B . Since the group $U(n)$ acts on orthonormal complex frames freely and transitively, E is a principal $U(n)$ -bundle. Consider the standard $\mathbb{C}^n$ -representation V of $U(n)$. By construction, the vector bundle $(E \times V)/U(n)$ coincides with B , hence this construction gives a reduction of the structure group of B to $U(n)$.

We have described the correspondence between the vector bundles (with appropriate reduction of the structure group) and the principal G -bundles. It turns out that this construction is well compatible with connections; one can define the holonomy and the curvature in both contexts, and these notions are equivalent. This is a part of standard differential geometry course, see e. g. [St, KN].

We will presently give a partial description of this correspondence.

Definition 2.6: Let $\pi : E \rightarrow M$ be a smooth fibration, with $T_\pi E$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\text{hor}}E \subset TE$ such that $TE = T_{\text{hor}}E \oplus T_\pi E$. The **parallel transport** along the path $\gamma : [0, a] \rightarrow M$ associated with the Ehresmann connection is a diffeomorphism $V_t : \pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(t))$ smoothly depending on $t \in [0, a]$ and satisfying $\frac{dV_t}{dt} \in T_{\text{hor}}E$.

Definition 2.7: Let B be a vector bundle on M and $\text{Tot } B \xrightarrow{\pi} M$ its total space. An Ehresmann connection on π is called **linear** if it is preserved by the homothety map $\text{Tot } B \rightarrow \text{Tot } B$ mapping v to λv and by the addition map $\text{Tot}(B \oplus B) \rightarrow \text{Tot } B$, that is, addition preserves horizontal vectors.

Proposition 2.8: A notion of a linear Ehresmann connection on a vector bundle B coincide with the usual notion of a connection; the corresponding parallel transport maps coincide as well.

Proof: [OV, Proposition 2.21]. ■

Definition 2.9: Let now $\pi : E \longrightarrow M$ be a principal G -bundle. A G -connection on π is a G -invariant Ehresmann connection.

Remark 2.10: Let $\pi : E \longrightarrow M$ be a principal G -bundle with G -connection ∇ , and X a smooth manifold with G -action ρ . Then the associated bundle $E_X := (E \times X)/G$ inherits an Ehresmann connection: $TE_X = T_{\text{hor}}E \oplus (TX/\rho(\mathfrak{g}))$, where $\mathfrak{g} = \text{Lie}(G)$ is the Lie algebra of G .

Definition 2.11: Let B be a vector bundle on M with structure group G , V representation of G , and E the corresponding principal G -bundle, $B = (E \times V)/G$. Then for any G -connection on E induces an Ehresmann connection on B as in Remark 2.10. Clearly, this connection is linear, hence induces a connection on B as on a vector bundle (Proposition 2.8). This connection is called **induced by the G -connection ∇^E** .

The holonomy of a G -connection and its induced connection are compatible, in the following sense:

Claim 2.12: Let E the a principal G -bundle on M , V a representation of G , and $B = (E \times V)/G$ the corresponding vector bundle with the structure group G . Let ∇^E a G -connection on E , and ∇ the induced connection on B . Denote the holonomy of ∇^E along a loop γ based on $m \in M$ by $g_\gamma \in G$. Then the holonomy of ∇ along γ is obtained from the action of g_γ on $B|_m = (E|_m \times V)/G$.

Proof: [KN]. ■

Throughout this paper, we use one of these equivalent languages, and assume tacitly an analogous statement for the other one. It is slightly more convenient to speak of non-commutative Barge-Ghys quasimorphisms in the language of vector bundles: this way, the correspondence with the usual, commutative Barge-Ghys quasimorphism is more apparent. However, it is more natural to state and prove the generalization of Kazhdan's theorem in the language of principal G -bundles. The translation from one of these languages to another is straightforward and is left to the reader as an exercise.

3 Barge-Ghys quasimorphisms on fundamental groups of closed manifolds of strictly negative curvature

3.1 Manifolds with strictly negative sectional curvature

In this preliminary section we list a few arguments of Riemannian geometry, most of them either classical or due to M. Gromov, [G2]. For manifolds of constant negative curvature, all the results we are going to obtain are classical and well known ([Fri, Chapter 8]); however, for arbitrary Riemannian manifolds of strictly negative curvature, a more subtle approach is required.

Theorem 3.1: (Cartan-Hadamard)

Let M be a complete, simply connected Riemannian manifold of non-positive sectional curvature. Then M is contractible.

Proof: We give a sketch of the proof, following [BBI]. In [BBI] this result was stated and proven for CAT(0)-spaces, but the comparison inequalities which are required by the CAT(0)-geometry easily follow from the non-positivity of the sectional curvature.

By Hopf-Rinow theorem, every two points of M can be connected by a geodesic. Let $\gamma_1 : [a, b] \rightarrow M$ and $\gamma_2 : [c, d] \rightarrow M$ be segments of geodesics in M , parametrized by the arc length. As shown in [BBI], the distance function $D : [a, b] \times [c, d] \rightarrow \mathbb{R}^{>0}$ taking x, y to $d(\gamma_1(x), \gamma_2(y))$ is strictly convex, unless the geodesics γ_1, γ_2 are segments of the same geodesic line. This implies, in particular, that any two points are connected by a unique geodesic: indeed, if γ_1 and γ_2 have the same ends, $\gamma_1(a) = \gamma_2(c)$ and $\gamma_1(b) = \gamma_2(d)$ the function D would be equal to 0 in (a, c) and (b, d) , hence it is zero on the diagonal, and the images of γ_1 and γ_2 coincide.

Fix a reference point $p \in M$ and consider the function

$$H : M \times [0, 1] \rightarrow M$$

taking $x \in M$ and $t \in [0, 1]$ to $\gamma(t \cdot d(p, x))$, where $\gamma : [0, d(p, x)]$ is the geodesic connecting p to x . A similar argument implies that H is continuous; clearly, H is a deformation retraction of M to p , hence M is contractible. ■

We will not use the Cartan-Hadamard theorem, but we use its corollary, which is inherent in its proof.

Corollary 3.2: Let M be a simply connected, complete manifold of non-positive sectional curvature. Fix a reference point $p \in M$ and consider the function $H_p : M \times [0, 1] \rightarrow M$ taking $x \in M$ and $t \in [0, 1]$ to $\gamma(t \cdot d(p, x))$, where $\gamma : [0, d(p, x)] \rightarrow M$ is the geodesic connecting p to x . Then H_p is continuous. ■

We could use the map H_p to define a **straight singular simplex**, following [G2]. Since we need only 2-dimensional simplices, we restrict ourselves to the 2-dimensional case.

Definition 3.3: Let M be a simply connected, complete manifold of non-positive sectional curvature. Let γ be the geodesic segment connecting b to c . We obtain a triangle by connecting a to all points of γ by a unique geodesic. The **geodesic simplex** $\Delta(a, b, c)$, associated with the points $a, b, c \in M$ is the union $\cup_{t \in [0, 1]} [H_a(t)(b), H_a(t)(c)]$, where $H_a(t) : M \rightarrow M$ is the homotopy along geodesics passing through a , defined in Corollary 3.2, and $[H_a(t)(b), H_a(t)(c)]$ the geodesic segment connecting $H_a(t)(b)$ and $H_a(t)(c)$.

Remark 3.4: The boundary of the simplex $\Delta(a, b, c)$ is the union of geodesics connecting a to b , b to c and c to a . Indeed, the homothety $H_a(t)$ moves any point $x \in M$ along the geodesic connecting x to a as t goes from 1 to 0, hence the segments of the boundary connecting b to a and c to a are geodesics; the third segment is γ , which is also chosen geodesic.

Remark 3.5: Note that the order of the points $a, b, c \in M$ is important. Indeed, unless M has constant sectional curvature, the simplex $\Delta(a, b, c)$ and (say) $\Delta(b, a, c)$ are different: otherwise, if $\Delta(a, b, c) = \Delta(b, a, c) = \Delta(a, c, b)$, this simplex is a segment of a completely geodesic 2-dimensional submanifold, and a general Riemannian manifold does not have completely geodesic submanifolds ([MW]).

The main technical result about manifolds of strictly negative sectional curvature which we use is the following theorem of Gromov, which is proven in [G2] for any straight singular simplex.

Theorem 3.6: Let M be a simply connected, complete manifold of strictly negative sectional curvature $K(M) < -\varepsilon < 0$, and $\Delta(a, b, c)$ a geodesic simplex defined above. Then the Riemannian area $\text{Area}(\Delta(a, b, c))$ satisfies $\text{Area}(\Delta(a, b, c)) \leq \pi \varepsilon^{-2}$.

Proof: [G2, §I.3]. ■

In the sequel, we will need two statements about uniqueness of geodesics on manifolds of strictly negative sectional curvature. The first is a direct consequence of the Cartan-Hadamard's theorem ([Theorem 3.1](#)).

Claim 3.7: Let $\gamma \in \pi_1(M, p)$ be an element of a fundamental group of a closed manifold of strictly negative sectional curvature. Then γ is represented by a unique geodesic loop based at p . ■

Another statement (slightly less trivial) deals with *free geodesic loops*. Recall that a **free geodesic loop** is an immersed submanifold of dimension 1 which is locally geodesic.

Proposition 3.8: Let M be a closed manifold of strictly negative sectional curvature, and $\varphi : S^1 \rightarrow M$ be a smooth map. Then there exists a unique free geodesic loop φ_1 which is free homotopic to φ . Moreover, φ_1 strictly minimizes the length of the loop.

Proof: [[K1](#), Theorem 3.8.14]. ■

3.2 Non-commutative Barge-Ghys quasimorphisms

Throughout this section, G is a connected Lie group.

Definition 3.9: Let M be a closed manifold with non-positive sectional curvature, and (P, ∇) a principal G -bundle with connection. Fix $x \in M$. The **non-commutative Barge-Ghys map** takes $\gamma \in \pi_1(M, x)$ to the holonomy of ∇ along the geodesic path starting and ending at x and homotopic to γ .¹

Remark 3.10: We defined the Barge-Ghys map for an arbitrary closed manifold M of non-positive sectional curvature and an arbitrary Lie group G . However, for all applications made in the present paper, M has strictly negative curvature, $G = A \rtimes K$ a Lie group satisfying (1.2), and ∇ a connection with the curvature 2-form taking values in the Lie algebra of K .

Remark 3.11: Throughout this paper, we consider the maps $q : \Gamma \rightarrow G$, where $\Gamma = \pi_1(M)$ is the fundamental group of a closed manifold of strictly negative curvature. Such groups are Gromov hyperbolic, however, it is not

¹Since M has non-positive curvature, this geodesic path is unique in its homotopy class, [Proposition 3.8](#).

clear yet whether our constructions can be generalized to all Gromov hyperbolic groups. Nevertheless, all results we prove remain valid whenever each geodesic simplex in the universal cover of M can be filled by a disk of bounded area, as in [Theorem 3.6](#). Note that the homological version of the bounded filling result is also true for hyperbolic groups which are not of geometric origin ([\[M\]](#)).

Theorem 3.12: Let M be a closed manifold of strictly negative sectional curvature, and Θ a geodesic n -polygon in M , that is, a contractible loop of n geodesic segments. Consider a principal G -bundle (P, ∇) with connection on M , and let $h(\Theta) \in G$ be the holonomy along the boundary of Θ , considered as a loop starting and ending at $p \in \Theta$. Assume that $G = A \rtimes K$ is a semidirect product of a Lie group A with a compact Lie group K , and the curvature 2-form of ∇ takes values in the Lie algebra $\text{Lie}(G)$ of G . Then $h(\Theta)$ belongs to a compact $C_n \subset G$ which is the same for all n -polygons Θ , but depends on n , (P, ∇) , and the bound on the curvature of M .

Proof: By the effective version of the Ambrose-Singer theorem, the holonomy along a path is linearly expressed in terms of the integral of the curvature over a disk filling this path ([\[RW, Theorem 1\]](#), [\[MO1\]](#), [\[Y\]](#)); before the integration, the curvature is transported to the origin using the holonomy of ∇ . Assume that for any path γ in M , the matrix norm of the corresponding holonomy map h_γ acting on the Lie algebra $\text{Lie}(G)$ is bounded by a constant $\tilde{C}$ independent from γ . Consider a contractible path γ filled by a disk Δ . Then the holonomy h_γ is bounded in terms of $\tilde{C}$, supremum of the curvature Θ of ∇ on Δ , and the Riemannian area of Δ . The area of any geodesic simplex is bounded by [Theorem 3.6](#). The absolute value of the curvature of ∇ is bounded from above because M is closed, and the curvature form on the pullback bundle $(\tilde{P}, \tilde{\nabla})$ is obtained as a pullback of the curvature of (P, ∇) .

Consider now the case when $G = A \rtimes K$ and the curvature takes values in $\text{Lie}(K)$. Since the Killing metric in $\text{Lie}(K)$ is holonomy invariant, the holonomy transports an element of $\text{Lie}(K)$ to the element of the same length. This implies existence of the constant $\tilde{C}$ defined earlier in this proof (indeed, $\tilde{C} = 1$, because the holonomy is an isometry). Since a geodesic 2-simplex on M has bounded area ([Theorem 3.6](#)), this implies that Hol_γ belongs to a fixed compact $C \subset G = A \rtimes K$ when $n = 3$ and Θ is a simplex.

When $n > 3$, we represent Θ as a boundary of the union of $n - 2$ geodesic simplexes $D_1, \dots, D_{n-2}$ with common vertex p . Then the holonomy $h(\Theta)$ is

obtained as a product $h(\Theta) = h(D_1)h(D_2)\dots h(D_{n-2}) \subset C^{n-2}$. Therefore, $h(\Theta)$ belongs to a fixed compact $C_n := C^{n-2}$, independent from the choice of Θ . ■

Remark 3.13: This innocuous statement, based on effective Ambrose-Singer theorem, has subtle depth, ignored in [BV, Theorem 3.11], which was the source of a grave error. In [BV, Theorem 3.11], we ignored the holonomy action on the Lie algebra of G , and came up with the false conclusion that the holonomy along the boundary of a contractible geodesic simplex is bounded, that is, belongs to a compact subset of G . This statement is in fact true, under an additional assumption that the holonomy action on a Lie algebra $\mathfrak{g} := \text{Lie}(G)$ of G preserves a metric on $\mathfrak{g}$. However, the action of the holonomy group (which itself belongs to G) on the Lie algebra of G is factorized through the adjoint action of G on $\mathfrak{g}$. The latter is an isometry (for some metric) only if G is, up to a finite covering, isomorphic to a product of an abelian group and a compact Lie group.

If we restrict ourselves to G with the adjoint action preserving a metric on $\mathfrak{g}$, then (up to a finite covering) $G = A \times K$, where A is abelian and K compact. This way we obtain a Barge-Ghys map which is equivalent to a quasimorphism taking values in A (we call such quasimorphisms “commutative”, Subsection 1.3; it is a classical notion).

However, if the curvature takes values in a Lie subalgebra $\mathfrak{k} \subset \text{Lie}(G)$, and the holonomy acts on $\mathfrak{k}$ by isometries, Theorem 3.12 remains valid, and we obtain many new quasimorphisms with values in G . This is the main subject of the present paper.

Theorem 3.14: Let M be a closed manifold of strictly negative sectional curvature, and (P, ∇) a principal G -bundle with connection. Assume that $G = A \rtimes K$, and the curvature 2-form takes values in the Lie algebra of K , as in (1.2). Fix $x \in M$, and let $q : \pi_1(M) \rightarrow G$ be the non-commutative Barge-Ghys map associated with (P, ∇) . Then q is an Ulam quasimorphism (Definition 1.1).

Proof: Denote by $\widetilde{M} \xrightarrow{\pi} M$ the universal cover of M . Let $a, b \in \pi_1(M)$, and $P_a, P_b \in \text{Diff}(\widetilde{M})$ the corresponding deck transformations. Fix a preimage $\tilde{x} \in \pi^{-1}(x)$, and denote by $(\tilde{P}, \tilde{\nabla})$ the pullback of (P, ∇) to $\widetilde{M}$. By definition, the product $q(ab)q(b)^{-1}q(a)^{-1}$ is represented by the holonomy of $(\tilde{P}, \tilde{\nabla})$ along the geodesic simplex connecting the points $\tilde{x}, P_a(\tilde{x})$, and

$P_b(P_a(\tilde{x}))$ in $\widetilde{M}$. By [Theorem 3.12](#), this quantity belongs to a compact subset independent from the choice of $x \in M$ and $a, b \in \pi_1(M)$. ■

Remark 3.15: The set of equivalence classes of non-commutative Barge-Ghys quasimorphisms is very big. As follows from [Proposition 3.19](#) below, the usual (“commutative”) Barge-Ghys quasimorphism is a special case of the quasimorphism taking values in a Lie group as defined in [Definition 3.9](#). By [\[BG\]](#), [\[C\]](#), the vector space spanned by commutative Barge-Ghys quasimorphisms up to equivalence is infinite-dimensional; in [Theorem 4.14](#) we construct non-commutative Barge-Ghys quasimorphisms which cannot be obtained from the commutative ones.

Remark 3.16: Using the Riemann-Hilbert correspondence ([\[OV, §2.8\]](#)) one can associate a flat principal G -bundle over M with each group homomorphism $\rho: \pi_1(M) \rightarrow G$. By construction, the holonomy of this flat connection is equal to ρ ([\[OV, §2.50\]](#)). Therefore, the corresponding Barge-Ghys quasimorphism is equal to ρ . In other words, all group homomorphisms to a Lie group can be realized as Barge-Ghys quasimorphisms.

3.3 Commutative and non-commutative Barge-Ghys quasimorphisms

Definition 3.17: Let M be a closed manifold of strictly negative sectional curvature, and $\theta \in \Lambda^1 M$ a 1-form on M . The **(commutative) Barge-Ghys quasimorphism** ([\[BG\]](#)) associated with θ takes $\gamma \in \pi_1(M)$ to the integral of θ over the geodesic path starting and ending at x and homotopic to γ .

Remark 3.18: Note that in all literature on quasimorphisms, one uses the additive notation: $|q(xy) - q(x) - q(y)| < C$. For the non-commutative quasimorphisms, we are forced to use the multiplicative notation,

$$q(xy)q(y)^{-1}q(x)^{-1} \in C.$$

When B is an oriented real rank-1 vector bundle on M , the commutative Barge-Ghys quasimorphism is actually equal to the “non-commutative Barge-Ghys quasimorphism” defined in [Definition 3.9](#); we explain this equivalence below.

Topologically, a rank 1 real oriented line bundle B is always trivial. To trivialize B , we choose a metric on B using the partition of unity, and trivialize B by taking a positive length-1 section u . Fix the connection ∇_0

on B in such a way that $\nabla_0 u = 0$. Then any connection ∇ on B can be written as $\nabla = \nabla_0 + \theta$, where $\theta \in \Lambda^1(M)$ is a 1-form. The holonomy of this connection along a loop γ is given by $\text{Hol}_\gamma(\nabla) = e^{-\int_\gamma \theta}$. Indeed, for any section $u_f := f(t)u$ of B restricted to a geodesic segment γ parametrized by $t \in [a, b]$, the equation

$$\nabla(u_f) = \frac{df}{dt}u + \theta f u = 0$$

is equivalent to $f' = -f\theta$, equivalently, $\frac{d \log f}{dt} = -\theta$.

Proposition 3.19: Let (B, ∇_0) be a trivial real rank 1 vector bundle on a closed manifold M of strictly negative sectional curvature and θ a 1-form on M . Consider the connection $\nabla := \nabla_0 - \theta$. Then the commutative Barge-Ghys quasimorphism associated with θ can be obtained as the logarithm of the “non-commutative” Barge-Ghys quasimorphism associated with (B, ∇) .

Proof: Indeed, for any loop in M , one has $\text{Hol}_\gamma(\nabla) = e^{-\int_\gamma \theta}$. ■

Remark 3.20: For any trivialized real rank 1 vector bundle B on M , the connections on B are in bijective correspondence with 1-forms on M . Therefore, Proposition 3.19 defines a bijective correspondence between the set of (non-commutative) Barge-Ghys quasimorphisms associated with B and the set of commutative Barge-Ghys quasimorphisms.

3.4 Translation length in Gromov hyperbolic groups

We proceed with a few observations about Gromov hyperbolic groups, used in the sequel.

Let Γ be a group generated by a finite set $\mathcal{A}$, and $C_\mathcal{A}(\Gamma)$ its Cayley graph. **Algebraic translation length** $\tau_\mathcal{A}(\gamma)$ of an element $\gamma \in \Gamma$ is defined ([BH]) as

$$\tau_\mathcal{A}(\gamma) := \lim_{n \rightarrow \infty} \frac{1}{n} d(1_\Gamma, \gamma^n).$$

The limit exists because the function $n \mapsto d(1_\Gamma, \gamma^n)$ is subadditive. It is not hard to see that the map $\gamma \rightarrow \tau_\mathcal{A}(\gamma)$ is conjugation invariant ([BH, Remark 1.3.14 (1)]). Further on, we shall use the following result, also found in [BH].

Proposition 3.21: Let Γ be a finitely generated Gromov hyperbolic group, and S_R the set of all conjugacy classes of all $\gamma \in \Gamma$ satisfying $\tau_\mathcal{A}(\gamma) < R$, where $R > 0$ is a real number. Then S_R is finite.

Proof: [BH, Proposition $\Gamma.3.15$]. ■

An element $\gamma \in \Gamma$ is called **primitive** if it cannot be represented as a power $\gamma = \varphi^n$, for any $n > 1$.

Corollary 3.22: Let Γ be a finitely generated Gromov hyperbolic group. Then any non-torsion element of Γ is an integer power of a primitive element.

Proof: By [C, Corollary 3.3.5], $\tau_{\mathcal{A}}(\gamma) > 0$ for all non-torsion γ . Clearly, $\tau_{\mathcal{A}}(\gamma^k) = k\tau_{\mathcal{A}}(\gamma)$. By Proposition 3.21, there exists a number $C \in \mathbb{R}^{>0}$ such that for all non-torsion $u \in \Gamma$ we have $\tau_{\mathcal{A}}(u) > C$. Then γ cannot be represented as n -th power for any $n > C^{-1}\tau_{\mathcal{A}}(\gamma)$. Let m be a maximal number such that γ is an m -th power of an element $\gamma_1 \in \Gamma$. Then γ_1 is primitive.² ■

3.5 HBG-quasimorphism associated with a connection

Definition 3.23: A quasimorphism $q : \Gamma \rightarrow G$ is called **homogeneous** if its restriction to any cyclic subgroup of Γ is a group homomorphism.

Let M be a closed Riemannian manifold of strictly negative sectional curvature, $\Gamma := \pi_1(M)$. By [P, Corollary 6.2.4], Γ is torsion-free. By Corollary 3.22, every non-torsion element of $\pi_1(M)$ is a power of a primitive element, which is unique by [BB, Lemma 2.2].

Given a primitive $\gamma \in \pi_1(M)$, let F_γ be the shortest free geodesic loop representing γ . By Proposition 3.8, F_γ is unique in its free homotopy class. Clearly, the conjugate elements of $\pi_1(M)$ correspond to the same free homotopy class. For each conjugacy class of γ we fix a choice of a point $x \in F_\gamma$. When $\gamma = \gamma_1^d$ and γ_1 is primitive, we denote by F_γ the loop F_{γ_1} iterated d times.

Let (B, ∇) be a bundle with connection, associated with a principal bundle which satisfies (1.2). Fix a point $p \in M$, and $\gamma \in \pi_1(M)$. Assume that the holonomy of the connection ∇ is $G = A \rtimes K$. We are going to define a homogeneous quasimorphism $q : \Gamma \rightarrow G \subset GL(B_p)$, where B_p denotes the fiber of B in p , as follows.

Consider a point $x \in F_\gamma$, and let $\tilde{F}_\gamma := \nu_{x,\gamma} \circ F_\gamma \circ \nu_{x,\gamma}^{-1}$ be the 3-segment piecewise geodesic path obtained by connecting p to x , going around the loop F_γ starting and ending in x , and going back to p along $\nu_{x,\gamma}$ in the

²We are grateful to Yves Cornulier and Sam Nead (<https://mathoverflow.net/users/1650/sam-nead>) who suggested a proof of this statement on Mathoverflow, [MO2].

opposite direction. Clearly, this path represents γ in $\pi_1(M, p)$. Denote by $q(\gamma) \in GL(B_p)$ the holonomy along $\tilde{F}_\gamma$. By construction, q restricted to a cyclic subgroup is always a homomorphism.

Note that q depends on the choice of $x \in F_\gamma$, which has to be fixed for each conjugacy class of $\gamma \in \Gamma$.

Theorem 3.24: Let M be a closed manifold with strictly negative sectional curvature, $p \in M$ a base point, $\Gamma := \pi_1(M)$, and $q : \Gamma \rightarrow G \subset GL(B_p)$ the map defined above, where G and ∇ satisfy (1.2). Then $q : \Gamma \rightarrow GL(B_p)$ is an Ulam quasimorphism. Moreover, q is homogeneous.

Proof: Let $\alpha, \beta, \gamma = (\alpha\beta)^{-1}$ be elements of Γ . Choose any points $a \in F_\alpha$, $b \in F_\beta$, $c \in F_\gamma$. Then $q(\alpha)q(\beta)q(\alpha\beta)^{-1}$ is a holonomy along a contractible geodesic polygon with 9 edges obtained by going along

$$\nu_{a,\alpha}, F_\alpha, \nu_{a,\alpha}^{-1}, \nu_{b,\beta}, F_\beta, \nu_{b,\beta}^{-1}, \nu_{c,\gamma}, F_\gamma, \nu_{c,\gamma}^{-1},$$

see Figure 1. However, the holonomy along any contractible geodesic polygon is bounded by Theorem 3.12. Homogeneity of q is clear, because $q(\gamma^n)$ is the holonomy of ∇ along the loop $\nu_{x,\gamma} \circ F_\gamma^n \circ \nu_{x,\gamma}^{-1}$. ■

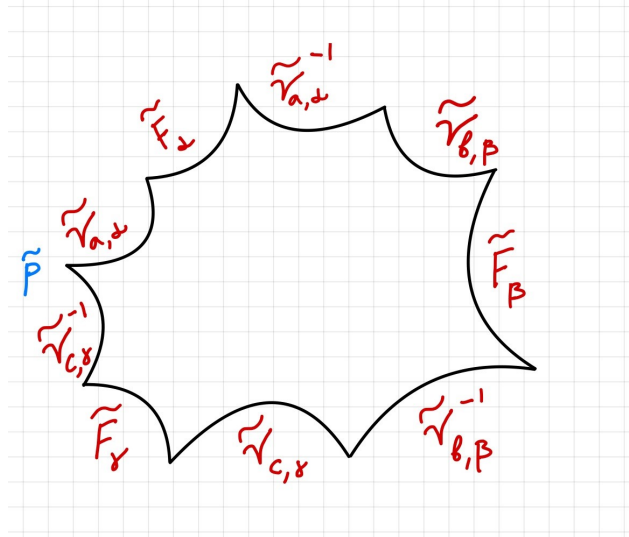


Figure 1: Lift of the geodesic polygon to the universal cover.

Definition 3.25: Let (B, ∇) be a vector bundle with connection, associated with a principal bundle satisfying (1.2), and q the Ulam quasimorphism defined in Theorem 3.24. Then $q : \Gamma \rightarrow GL(B_p)$ is called **the HBG-quasimorphism associated with (B, ∇)** .

Remark 3.26: Instead of vector bundles, Theorem 3.24 can be stated in the setting of principal bundles. In Section 2 we explained how it is done. This allows one to modify Definition 3.25 obtaining a HBG-quasimorphism associated with a principal G -bundle with connection, for a Lie group G satisfying (1.2).

Remark 3.27: Let $G = A \rtimes K$ be a group satisfying (1.2). For any homogeneous quasimorphism $q : \Gamma \rightarrow G$ and $r \in G$, the map $rq r^{-1}$ is also a homogeneous quasimorphism. The HBG-quasimorphism $q : \pi_1(M, p) \rightarrow G$ associated with a G -bundle depends on the choice of the points $x_i \in F_{\gamma_i}$ on the shortest loop representing the primitive element γ_i . However, this dependence is easy to describe: for any pair $x_i, y_i \in F_{\gamma_i}$, consider the geodesic simplex with vertices x_i, y_i, p , with the geodesic from x_i to y_i going in the same direction as in the loop F_{γ_i} . Let r_i denote the loop along this simplex, starting in p , going to x , then to y and to p again. Then $q(\gamma_i)$ is the holonomy along $\nu_{x_i, \gamma_i} F_{\gamma_i} \nu_{x_i, \gamma_i}^{-1}$, or, equivalently, the holonomy along $r_i \nu_{y_i, \gamma_i} F_{\gamma_i} \nu_{y_i, \gamma_i}^{-1} r_i^{-1}$. In other words, replacing the choice of x_i by y_i is equivalent to the conjugation of $q(\gamma_i)$ with the holonomy along a geodesic simplex r_i , which is bounded. These two quasimorphisms are Ulam equivalent because they are equivalent to the original Barge-Ghys quasimorphism (Claim 3.30).

In the commutative case, any homogeneous quasimorphism is conjugation invariant ([C, §2.2.3]). The following result is a non-commutative Barge-Ghys analogue of this statement.

Claim 3.28: Let $q : \Gamma \rightarrow G$ be a HBG-quasimorphism, and $\alpha, \beta \in \Gamma$ conjugate elements. Then $q(\alpha)$ is conjugate to $q(\beta)$.³

Proof: Let F_γ be the free geodesic loop homotopy equivalent to α and β , and $x \in F_\gamma$ be the chosen point. Then $q(\alpha)$, which is equal to holonomy of ∇ along the loop $\nu_{x, \alpha} \circ F_\gamma \circ \nu_{x, \alpha}^{-1}$, is conjugate to $q(\beta)$, obtained the same way from the loop $\nu_{x, \beta} \circ F_\gamma \circ \nu_{x, \beta}^{-1}$. It is easy to write this conjugation explicitly: $q(\alpha) = Rq(\beta)R^{-1}$, where R is holonomy of ∇ along the loop $\nu_{x, \alpha} \circ \nu_{x, \beta}^{-1}$. ■

³Clearly, the conjugating element depends on α and β .

Remark 3.29: When $G = \mathbb{R}$, $q(\alpha) = q(\beta)$ for α, β conjugate is true by [C, §2.2.3]. However, when G contains a central compact subgroup S , it is easy to construct a homogeneous Ulam quasimorphism $q : \Gamma \rightarrow G$ which does not have this property. For example, let $\Gamma = \mathbb{F}_2$ be a free group, and define an arbitrary map $q : \mathfrak{P} \rightarrow S$, where $\mathfrak{P}$ is the set of all primitive words in $\mathbb{F}_2$. If $q(W) = q(W^{-1})^{-1}$, this map extends to a homogeneous Ulam quasimorphism, and all homogeneous Ulam quasimorphisms $q : \mathbb{F}_2 \rightarrow S$, are obtained this way. Clearly, this map is not necessarily invariant under the conjugation. This implies, in particular, that Claim 3.28 is false for a general (not HGB) Ulam quasimorphisms, if G has a compact central subgroup.

Any commutative quasimorphism is equivalent to its homogenization, which is unique in its equivalence class ([C, §2.2.3]). For Ulam quasimorphisms (even those which take values in a commutative group), this is false, which can be seen from Remark 3.29.

Claim 3.30: Let (B, ∇) be a principal G -bundle with connection over a manifold M , or a vector bundle associated with G . Assume that (G, ∇) satisfies assumptions of (1.2), and M is a closed manifold of strictly negative sectional curvature. Let $\Gamma := \pi_1(M, p)$, and $q_\nabla : \Gamma \rightarrow G$ be the corresponding Barge-Ghys quasimorphism. After making the relevant choices, we obtain the HBG-quasimorphism $q : \Gamma \rightarrow G$. Then q is equivalent to q_∇ , and, moreover, $q(\gamma)q_\nabla(\gamma)^{-1} \in C_4$, where $C_4 \subset G$ is the compact subset defined in Theorem 3.12, which depends only on M , B and ∇ .

Proof: Let $\gamma \in \Gamma$ be any element, which is represented as $\gamma = \gamma_0^n$, where γ_0 is primitive, and let F_{γ_0} be the free geodesic representing the same free homotopy class. Fix a geodesic segment $\nu_{\gamma_0, x}$ connecting $p \in M$ to a point x in F_{γ_0} , and let $\tilde{\gamma}$ the geodesic segment connecting p to itself and homotopic to γ . Then $q_\nabla(\gamma)$ is a holonomy of ∇ along $\tilde{\gamma}$, and $q(\gamma)$ is the holonomy of ∇ along the geodesic chain $\nu_{\gamma_0, x} \circ F_{\gamma_0}^n \circ \nu_{\gamma_0, x}^{-1}$. Geometrically, the segment $F_{\gamma_0}^n$ is one geodesic interval, and therefore the polygon $\nu_{\gamma_0, x} \circ F_{\gamma_0}^n \circ \nu_{\gamma_0, x}^{-1} \circ \tilde{\gamma}^{-1}$ has 4 geodesic sides. By Theorem 3.12, the holonomy along the sides this polygon belongs to C_4 . ■

4 Constructible Ulam quasimorphisms

4.1 Virtually conjugation equivalent cyclic subgroups

In the sequel, we use the following notion.

Definition 4.1: Let $A, B \subset G$ be infinite cyclic subgroups of a group G . We say that A is **virtually conjugation equivalent (VCE equivalent)** to B , denoted $A \sim_{\text{vce}} B$, if there exist $u \in G$ such that the intersection $A \cap B^u$ is infinite, where B^u denotes the subgroup obtained from B by conjugation with u . We also write $x \sim_{\text{vce}} y$ for elements $x, y \in G$ when x, y generate infinite cyclic subgroups which are VCE equivalent.

Remark 4.2: In [DGO], two elements of a group are called **commensurable** if non-zero powers of these elements are conjugate. This is equivalent to VCE equivalence of cyclic subgroups generated by these elements.

Claim 4.3: The relation $\sim_{\text{vce}}$ is, indeed, an equivalence relation.

Proof: Let A, B, C be three cyclic subgroups which satisfy $A \sim_{\text{vce}} B$ and $B \sim_{\text{vce}} C$. Then A is commensurable with B^u and B is commensurable with C^v , that is, the intersections $A \cap B^u$ and $B \cap C^v$ are infinite. Note that any infinite subgroup in a cyclic group has finite index, hence to show that $A \sim_{\text{vce}} C$ it would suffice to show that $A \cap C^{vu}$ is infinite.

By definition, $C^{vu} \cap B^u$ is a finite index subgroup in B^u . On the other hand, B^u contains $A \cap B^u$ as a finite index subgroup. Since an intersection of two infinite subgroups in $\mathbb{Z}$ is always infinite, $A \cap B^u \cap C^{vu}$ is infinite, proving the claim. ■

Remark 4.4: If $\text{rk } H^1(\Gamma, \mathbb{Q}) \geq 2$, it is easy to find infinitely many cyclic subgroups which are pairwise VCE non-equivalent. Indeed, if the line in $H^1(\Gamma, \mathbb{Q})$ generated by a cyclic subgroup A is not collinear with the line generated by B , this implies that $A \not\sim_{\text{vce}} B$. To extend this statement to more general hyperbolic groups, we need the following argument.

The notion of hyperbolic embedding was defined in [DGO]. The following result is one of the applications of hyperbolic embeddings

Claim 4.5: Let H be hyperbolically embedded subgroup of Γ , then every (commutative) quasimorphism on H extends to a quasimorphism on Γ .

Proof: The quasimorphisms from H to $\mathbb{R}$ are controlled by kernel of the

map $H_b^2(H, \mathbb{R}) \longrightarrow H^2(H, \mathbb{R})$, called **the exact reduced bounded cohomology**, which is clear from (1.1). Every class in second exact reduced bounded cohomology of H can be extended to $H_b^2(\Gamma, \mathbb{R})$, as follows from [HO]. This result was generalized to all cohomology groups in [FPS]. ■

In [O], D. Osin defined acylindrically hyperbolic groups, which includes all hyperbolic groups which are not virtually cyclic. The fundamental group of a closed manifold of strictly negative sectional curvature, which is by convention assumed of dimension > 1 , is hyperbolic and not virtually cyclic, hence it is acylindrically hyperbolic. In [O, Theorem 1.2] and [DGO, Theorem 2.24] it was shown that any acylindrically hyperbolic group Γ contains a hyperbolically embedded subgroup $H = \mathbb{F}_2 \times K$, where K is some finite group. This brings the following theorem.

Theorem 4.6: Let Γ be an acylindrically hyperbolic group (this includes fundamental groups of a closed manifold of strictly negative sectional curvature.) Then there exists a free subgroup $H = \mathbb{F}_2 \subset \Gamma$ such that any quasimorphism on H can be extended to a quasimorphism on Γ . ■

Remark 4.7: Brooks quasimorphisms are quasimorphisms on a free group, defined in [Br]; see [C] for more details. We use Theorem 4.6 only Brooks quasimorphisms; however, for Brooks quasimorphisms this result is implicit already in Bestvina-Fujiwara [BF], where the free group is a Schottky subgroup.

The following result immediately follows from the definition of Brooks quasimorphism.

Claim 4.8: Let W_1 and W_2 be two reduced words, considered as elements of $\mathbb{F}_2$. Denote by $q_1 : \mathbb{F}_2 \longrightarrow \mathbb{R}$ the homogenization of the Brooks quasimorphism associated with W_1 . Assume that W_1 is not a subword of W_2^i , for all $i \in \mathbb{Z}$. Then $q_1(W_2) = 0$.

Proof: The proof is left as an exercise to the reader. ■

This implies the following proposition which will be used in the sequel.

Proposition 4.9: Let Γ be an acylindrically hyperbolic group.¹ Let k, k_1, l, l_1

¹This includes fundamental groups of a closed manifold of strictly negative sectional curvature.

be non-negative integers such that neither $k = 0, k_1 = 0$ nor $l = 0, l_1 = 0$, nor $k = k_1, l = l_1$ holds. Then there exists a free subgroup $H = \mathbb{F}_2 \subset \Gamma$ generated by $a, b \in \Gamma$ such that $a^k b^l \not\sim_{\text{vce}} a^{k_1} b^{l_1}$, where the relation $\sim_{\text{vce}}$ is taken in Γ .

Proof: By [Theorem 4.6](#), there exists a subgroup $\mathbb{F}_2 \subset \Gamma$ such that any quasimorphism is extended from $\mathbb{F}_2$ to Γ . Let $W_1 = a^{k_1} b^{l_1}$ and $W_2 = a^k b^l$. The negation of the condition “ $k = 0, k_1 = 0$ or $l = 0, l_1 = 0$, or $k = k_1, l = l_1$ ” is equivalent to “ W_1 is not a subword of W_2^i , and W_2 is not a subword of W_1^i , for all $i \in \mathbb{Z}$.” Let $q_1, q_2 : \mathbb{F}_2 \rightarrow \mathbb{R}$ be the homogenizations of the corresponding Brooks quasimorphisms. By construction, q_i can be extended to quasimorphisms $q_1, q_2 : \Gamma \rightarrow \mathbb{R}$. Since homogeneous quasimorphisms are conjugate invariant, and $q_i(W_j) = \delta_{ij}$, the cyclic group $\langle W_1 \rangle$ intersects trivially with any conjugate to $\langle W_2 \rangle$. ■

As an immediate corollary, we obtain the following assertion, which is also implied by [\[BF\]](#).²

Corollary 4.10: Let Γ be an acylindrical hyperbolic group. Then there exists infinitely many cyclic subgroups U_i which satisfy $U_i \not\sim_{\text{vce}} U_j$ for any $i \neq j$. ■

Remark 4.11: In [\[DGO, Corollary 6.12\]](#), a similar result is shown: the authors construct an arbitrarily large collection of elements in a hyperbolic group which are non-commensurable in the sense of [Remark 4.2](#).

4.2 Non-constructible Ulam quasimorphisms

In [\[FK\]](#), Kapovich and Fujiwara define a constructible quasimorphism taking values in a discrete group. We extend their definition to any Lie group.

Definition 4.12: Let $q : \Gamma \rightarrow G$ be an Ulam quasimorphism to a Lie group G . Consider a finite index subgroup $\Gamma_0 \subset \Gamma$, and let $H \supset q(\Gamma_0)$ be a subgroup of G containing $q(\Gamma_0)$. Assume that H contains a normal subgroup N , abelian and central in H , such that the composition $\Gamma_0 \xrightarrow{q} H \rightarrow H/N$ is equivalent to a homomorphism $\Gamma_0 \rightarrow H/N$. Then q is called a **constructible quasimorphism**.

²For non-elementary hyperbolic groups, this result also follows from [\[EF\]](#).

An element $a \in \Gamma$ of a group is called **primitive** if for any $n \in \mathbb{Z}$ and $b \in \Gamma$ such that $b^n = a$, one has $n = \pm 1$. A quasimorphism $q : \Gamma \rightarrow G$ is called **homogeneous** if its restriction to any cyclic subgroup is a homomorphism (Definition 3.23). We gave a detailed treatment of homogeneous quasimorphisms in Subsection 3.5, where we defined a new type of quasimorphisms, called HBG-quasimorphism, for “homogeneous Barge-Ghys”. We use the acronym HBG to avoid the confusion, because the HBG-quasimorphisms are not, in fact, “Barge-Ghys quasimorphisms”, in the sense of Definition 3.9. One should think of a HBG-quasimorphism as of a “homogenization” of a Barge-Ghys quasimorphism. However, the notion of homogenization, which is well known for the usual ($\mathbb{R}$ -valued) quasimorphisms, is not defined for the quasimorphisms taking values in a Lie group.

In Subsection 5.2, we prove the following theorem.

Theorem 4.13: Let M be a closed manifold of strictly negative curvature, $G = A \rtimes K$ a Lie group satisfying (1.2), and $x_1, \dots, x_n \in \Gamma := \pi_1(M)$ a collection of primitive elements satisfying $x_i \not\sim_{\text{vce}} x_j$ for $i \neq j$.³ Let $A_1 \subset A$ be a normal subgroup generated by the commutators $aka^{-1}k^{-1}$, where $a \in A, k \in K$. Fix $a_1, \dots, a_n \in A_1, b_1, \dots, b_n \in K$. Then there exists a connection ∇ satisfying (1.2) on a trivial principal bundle P such that the corresponding HBG-quasimorphism $q_\nabla : \Gamma \rightarrow G$ takes x_i to (a_i, b_i) , $i = 1, 2, \dots, n$.

Proof: See Subsection 5.2 below. ■

We use this result to prove Theorem 4.14 below.

Theorem 4.14: Let $G = A \rtimes K$ be a Lie group satisfying (1.2), and $\Gamma := \pi_1(M)$, where M is a closed manifold of strictly negative sectional curvature, $\dim_{\mathbb{R}} M > 1$. Assume that A is isomorphic to $\mathbb{R}^n$, and K acts on A with all orbits dense in a sphere in $A = \mathbb{R}^n$.⁴ Assume also that K has trivial center. Then there exists a non-constructible HBG-quasimorphism $q_\nabla : \Gamma \rightarrow G$.

Proof. Step 1: Let $\kappa_1, \dots, \kappa_n$ be a collection of elements of K which satisfy the following condition: for any i and j , the cyclic subgroup of K generated by κ_i^j is dense in a positive-dimensional abelian subgroup $\hat{T}_i(j)$. We can always assume that the connected component T_i of $\hat{T}_i(j)$ is independent

³Since $\dim M > 1$, there exist infinitely many such subgroups, as follows from Corollary 4.10.

⁴This condition is stronger than irreducibility of the representation associated with the action of K on A .

from j (for example, take κ_i with dense orbit in a circle). We also assume that K is generated by all connected components $T_i \subset \hat{T}_i$, $i = 4, \dots, n$.

Choose elements $u, v \in \Gamma$ such that $u^l v^m \not\sim_{\text{vce}} u^{l_1} v^{m_1}$ unless $l = l_1, m = m_1$ or $l = l_1 = 0$ or $m = m_1 = 0$ (Proposition 4.9). This gives us infinitely many elements of Γ which pairwise satisfy $x_i \not\sim_{\text{vce}} x_j$ for $i \neq j$. Denote the identity element in G by 1_G . We apply Theorem 4.13 to $x_1 = u, x_2 = v, x_3 = u^l v^m$ such that $l, m \neq 0$, and $x_4, \dots, x_n$ any collection of elements of Γ which satisfies $x_i \not\sim_{\text{vce}} x_j$ for $i \neq j$. Using Theorem 4.13, we obtain a HBG-quasimorphism q_∇ such that $q_\nabla(u) = 1_G, q_\nabla(v) = 1_G$, and $q_\nabla(x_3) = (a, 1_K)$, where 1_K is the unity in K and $a \in A$ a non-zero vector, and $q_\nabla(x_i) = (1_A, \kappa_i)$ for $i \geq 4$.

We are going to show that q_∇ is not constructible.

Step 2: By contradiction, suppose that q_∇ is constructible, that is, there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ and $q_\nabla|_{\Gamma_0}$ satisfies the assumptions of Definition 4.12.

Since Γ_0 is of finite index in Γ , it contains powers $x_i^{n_i}$ of each x_i , for some $n_i \in \mathbb{Z}^{>0}$. Clearly, the closure of the union $\bigcup_{i \geq 4} \langle \kappa_i^{n_i} \rangle$ contains $\bigcup_{i \geq 4} T_i$, which generates K . This implies that the group H of Definition 4.12, containing $q(\Gamma_0)$, contains a dense subgroup of $\{1_A\} \rtimes K \subset G$. Since every K -orbit of a non-zero vector in A is dense in a sphere, every H -orbit of such a vector is dense in a sphere. This implies that the closure of H contains $S \times K \subset A \rtimes K$, where $S \subset A = \mathbb{R}^n$ is a sphere. Also, since K is simple and does not have centralizers in A , the center of G (and, therefore, of H) is trivial. This implies that the group N of Definition 4.12 is trivial.

Step 3: Let $q_1 : \Gamma_0 \rightarrow G$ denote the restriction of $q_\nabla : \Gamma \rightarrow G$ to Γ_0 . By Definition 4.12, there exists a homomorphism $h : \Gamma_0 \rightarrow G$ which is equivalent to q_1 . Then there exists a compact $C \subset G$ such that

$$h(z) \in C q_1(z) \quad (4.1)$$

for all $z \in \Gamma_0$.

Since Γ_0 has finite index in Γ , there exist $i, j > 0$ such that $u^i, v^j \in \Gamma_0$. Restricted to any cyclic subgroup in Γ , the quasimorphism q_i is a homomorphism, because it is homogeneous. Since $q_1(u^{ki}) = q_1(v^{kj}) = 1_G$ for all $k \in \mathbb{Z}$, this gives $h(u^{ik}) = h(u^i)^k \in C$ and $h(v^{jk}) = h(v^j)^k \in C$. This implies that $h(u), h(v)$ belong to a compact subgroup of G . Clearly, any $(a, k) \in G$ with a not equal to unity generates a non-bounded subgroup of G . Therefore, $h(u), h(v) \in 1_A \rtimes K$.

Step 4: Let $x_3 = u^l v^m$ be as above, and p be a positive integer such that $x_3^p \in \Gamma_0$. By construction, $q_\nabla(x_3) = (a, 1_K)$, where $a \neq 1_A$. Then $q_1|_{\langle x_3^p \rangle}$ is a homomorphism with unbounded image, hence $h|_{\langle x_3^p \rangle}$ cannot be bounded. In Step 3, we proved that $h((u^l v^m)^p) \in 1_A \rtimes K$, obtaining a contradiction. This implies that q_∇ is not constructible. ■

Remark 4.15: To simplify the arguments, we have proved [Theorem 4.14](#) in excessively restrictive assumptions. In fact the arguments used in [\[BV\]](#) can be used to prove a similar result for $G = A \rtimes K$ where A is a torsion-free nilpotent Lie group equipped with an appropriate K -action.

5 HBG-quasimorphisms with prescribed values

5.1 Connections with prescribed holonomy

We prove the following preliminary lemma, which will be used later in this section. This lemma is essentially a group-valued version of the Newton-Leibniz theorem.

Lemma 5.1: Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, and P a trivial G -bundle on an interval $[0, 1]$. Fix an element $g \in G$. Denote by ∇_0 the trivial connection on P . Then there exists a $\mathfrak{g}$ -valued 1-form θ with compact support, such that the holonomy $\text{Hol}(\nabla)$ of the connection $\nabla := \nabla_0 + \theta$ is equal to g .

Proof: Write θ as $a(t)dt$, where $a(t) \in \mathfrak{g}$ and dt is the standard 1-form on $[0, 1]$. Denote by $\gamma : [0, 1] \rightarrow G$ the holonomy of ∇ . Then the connection 1-form satisfies $a(t) = \gamma(t)^{-1}\dot{\gamma}(t)$. By Newton-Leibnitz formula, $\int_0^1 \dot{\gamma} dt = g$. Setting $a(t) := \gamma(t)^{-1}\dot{\gamma}(t)$, we obtain a connection form which satisfies $\text{Hol}(\nabla) = \int_0^1 \gamma(t)a(t)dt = g$. If we choose $\gamma(t)$ constant in a neighbourhood of 0 and of 1, the form $a(t)dt$ would have compact support. ■

When $G = A \rtimes K$ is a Lie group satisfying [\(1.2\)](#), we have a version of [Lemma 5.1](#) which is most appropriate for our constructions.

Lemma 5.2: Let $G = A \rtimes K$ be a Lie group which satisfies [\(1.2\)](#), and P a trivial G -bundle on an interval $[0, 1]$. We denote by $A_1 < A$ the subgroup generated by the commutators $aka^{-1}k^{-1}$, where $a \in A, k \in K$. Fix an

element $(a, k) \in A_1 \rtimes K \subset G$. Denote by ∇_0 the trivial connection on P . We denote the Lie algebra of G by $\mathfrak{g}$, decomposed as $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{k}$. Then there exists a $\mathfrak{g}$ -valued 1-form $\theta = \theta_a + \theta_k$, with θ_a taking values in $\mathfrak{a}$ and θ_k in $\mathfrak{k}$ such that the holonomy $\text{Hol}(\nabla)$ of the connection $\nabla := \nabla_0 + \theta$ on $[0, 1]$ is equal to (a, k) , and the holonomy of $\nabla_0 + \theta_a$ is trivial.

Proof: It suffices to prove [Lemma 5.2](#) for any collection of generators of $A_1 \rtimes K$; we prove it for $(a_1, 1_K)$, $a_1 = aka^{-1}k^{-1} \in A_1$ and for $(1_A, k)$. The second assertion follows from [Lemma 5.1](#) directly: any element of K can be obtained as a holonomy of a connection defined by a 1-form with coefficients in $\mathfrak{k}$. To obtain $(a_1, 1_K)$, we split our interval in 4 pieces, take an $\mathfrak{a}$ -valued 1-form on the first piece, and, using [Lemma 5.1](#), arrive at a after taking the parallel transport. Then we take a $\mathfrak{k}$ -valued 1-form in the second piece and arrive at ak , using [Lemma 5.1](#) again, and repeat the same procedure, obtaining a connection with holonomy $aka^{-1}k^{-1}$. We chose the connection 1-form on the third part inverse to the first part, in such a way that the composition of their holonomies is 1_A . Let θ denote the connection form obtained this way. The $\mathfrak{a}$ -part of θ is non-zero only in the first and third part of the interval, and the holonomy of the corresponding connection $\nabla_0 + \theta_a$ is trivial by construction. ■

5.2 Constructing the HBG-quasimorphisms

In this section, we prove [Theorem 4.13](#). We repeat its statement for convenience.

Theorem 4.13: Let M be a closed manifold of strictly negative curvature, $G = A \rtimes K$ a Lie group satisfying (1.2), and $x_1, \dots, x_n \in \Gamma := \pi_1(M)$ a collection of primitive elements satisfying $x_i \not\sim_{\text{vce}} x_j$ for $i \neq j$.¹ Let $A_1 < A$ be a subgroup generated by the commutators $aka^{-1}k^{-1}$, where $a \in A, k \in K$. Fix $a_1, \dots, a_n \in A_1$, $b_1, \dots, b_n \in K$. Then there exists a connection ∇ satisfying (1.2) on a trivial principal bundle P such that the corresponding HBG-quasimorphism $q_\nabla : \Gamma \rightarrow G$ takes x_i to (a_i, b_i) , $i = 1, 2, \dots, n$.

Proof. Step 1: For each primitive element $x \in \pi_1(M)$, denote by F_x the minimal free geodesic loop representing x ([Proposition 3.8](#)). This geodesic loop is unique and determines x up to conjugation. Fix a point p on M , and a point p_i on each F_{x_i} . Let γ_{x_i} be the piecewise smooth loop connecting p

¹Since $\dim M > 1$, there exist infinitely many such subgroups, as follows from [Corollary 4.10](#).

to p_i by a geodesic segment $\nu_{p_i, F_{x_i}}$, going around F_{x_i} , and back from p_i to p by the same geodesic segment $\nu_{p_i, F_{x_i}}$ reversed. For each connection ∇ on P , the corresponding HBG-quasimorphism takes x_i to the holonomy of ∇ along γ_{x_i} .

We are going to choose a connection ∇ on P such that the holonomy of ∇ along γ_{x_i} is equal to $g_i := (a_i, b_i)$.

Step 2: Since $x_i \not\sim_{\text{VCE}} x_j$, none of the loops $F_{x_1}, \dots, F_{x_n}$ is contained in another of these loops. This is actually the only reason why we care about the VCE equivalence. Fix an open set B_{x_i} containing a segment of F_{x_i} and not intersecting the rest of the loops. We can choose B_{x_i} in such a way that it does not intersect the geodesic segment connecting p to p_i (Step 1). Denote by ∇_0 the trivial connection on P . Using a form of Newton-Leibniz theorem ([Lemma 5.1](#)) we can construct a connection 1-form on each open set B_{x_i} in such a way that the holonomy of the corresponding connection along $\gamma_i \cap B_{x_i}$ is equal to any given element of G . However, this connection will not be flat on the first component of $G = A \rtimes K$, and we need it to be flat. This is why we are forced to use the more nuanced form of Newton-Leibniz theorem, namely [Lemma 5.2](#).

The difference between [Lemma 5.1](#) and [Lemma 5.2](#) is that the first produces a 1-form θ for a connection with prescribed holonomy, and we can then extend this 1-form to M . The second one produces a 1-form $\theta = \theta_a + \theta_k$ with prescribed holonomy and an extra restriction: the holonomy of $\nabla_0 + \theta_a$ is trivial.

By Newton-Leibniz theorem, this means that $\theta_a = g_t^{-1} \dot{g}_t$, where $g_t : [0, 1] \rightarrow A$ is a function with values in the Lie group A , satisfying $g_1 = 1_A$.

We extend g_t to a smooth A -valued function $g : M \rightarrow A$ with support in B_{x_i} ("support", for an A -valued function, is the closure of the set where this function takes value different from 1_A). Consider a 1-form $g^{-1}dg$ evaluated at $m \in M$ as $g^{-1}(m)D_m g$, where $D_m : T_m M \rightarrow T_{g(m)} G$ is the derivative of g in m . Then $\theta_a := g^{-1}dg$ is a 1-form with values in the Lie algebra of A . Let $\nabla := \nabla_0 + \theta_a$ be the corresponding connection on a trivial A -bundle, where ∇_0 is the trivial connection. Its curvature F is written as

$$\begin{aligned} F &= d\theta_a + \theta_a \wedge \theta_a = d(g^{-1}dg) + g^{-1}dg \wedge g^{-1}dg = \\ &= -g^{-1}dg \wedge g^{-1}dg + g^{-1}dg \wedge g^{-1}dg = 0, \end{aligned}$$

where the third equality follows from applying the chain rule to $d(gg^{-1}) = 0$. In other words, the connection $\nabla_0 + \theta_a$ is flat.

We extend θ_k from the interval to a 1-form (also denoted by θ_k) with

support in B_{x_i} . Consider the connection $\nabla_0 + \theta_a + \theta_k$ on P extending the connection we have constructed on γ_{x_i} . Then $\nabla_0 + \theta_a + \theta_k$ is flat on the A -part, hence its curvature takes values in the Lie algebra of K .

This gives a new connection on P , equal to the old one outside of B_{x_i} , with prescribed holonomy $\mathfrak{H}$ on F_{x_i} . The holonomy of this connection along $\nu_{p_i, F_{x_i}} \circ F_{x_i} \circ \nu_{p_i, F_{x_i}}^{-1}$ is $\mathfrak{S}\mathfrak{H}\mathfrak{S}^{-1}$, where $\mathfrak{S} : P|_p \rightarrow P|_{p_i}$ is the holonomy of this connection along $\nu_{p_i, F_{x_i}}$. Since the map $\mathfrak{H} \mapsto \mathfrak{S}\mathfrak{H}\mathfrak{S}^{-1}$ is bijective, by an appropriate choice of the connection on B_{x_i} , we can obtain any element of $A_1 \rtimes K$ as holonomy of the connection along $\nu_{p_i, F_{x_i}} \circ F_{x_i} \circ \nu_{p_i, F_{x_i}}^{-1}$.

Using a partition of unity, we can glue this 1-form to the connection form in ∇_0 , obtaining another connection which is equal to ∇_0 outside of B_{x_i} , and has prescribed holonomy on $\gamma_i \cap B_{x_i}$. This allows us to modify ∇_0 on each open set B_{x_i} in such a way that the holonomy of ∇ along $\gamma_i \cap B_{x_i}$ is equal to g_i .

We built a connection ∇ which satisfies [Theorem 4.13](#). ■

5.3 ε -representations

As an application of the construction given in Subsection 5.2, we generalize Kazhdan's theorem [[Kaz](#), Theorem 2] to the fundamental group of an arbitrary closed manifold of strictly negative sectional curvature. We define the notion of an ε -representation and a δ -approximated ε -representation as follows (compare with [[Kaz](#)] and Subsection 1.6).

Let G be a topological group. Recall that **Chabauty topology** on the set of closed subgroups $C(G)$ is defined by the base of neighbourhoods, $W_U(\Gamma)$ given for each open subset $U \subset G$ and a subgroup $\Gamma \subset G$

$$W_U(\Gamma) = \{\Gamma' \in C(G) \mid U \cdot \Gamma' \supset \Gamma \text{ and } U \cdot \Gamma \supset \Gamma'\}.$$

If the topology of G is induced by a left-invariant metric, the Chabauty topology is induced by the Hausdorff metric on the space of subsets of G .

By [[F](#), p. 474], or [[dH](#), Proposition 2 (vi)], the space $C(G)$ is compact.

Recall that a group G **has no small subgroups** ([[Tao](#), Exercise 1.5.6, Corollary 1.5.8]) if there exists a neighbourhood of identity which does not contain a non-trivial subgroup; this property is clearly satisfied by any Lie group. When G has no small subgroups, the space of $C_0(G)$ closed non-trivial subgroups is also compact.

Clearly, the diameter of a subset of a metric space M is a continuous function in the topology on 2^M induced by the Hausdorff metric. Since $C_0(G)$ is compact, the diameter of non-trivial subgroups in G is bounded from below for any metric on G .

Fix a left-invariant metric on G such that any non-trivial subgroup has diameter at least $1/3$. We motivate the choice of the constant $1/3$ as follows. For compact group G , we can always choose a bi-invariant metric; its geodesics are translations of one-parametric subgroups. We normalize the metric such that the diameter of any compact one-parametric subgroup is bounded from below by $1/2$.

For a bi-invariant Riemannian metric on a Lie group, the geodesics are obtained by translation of one-parametric group: the Lie-theoretic exponential map coincides with the Riemannian. Every compact Lie group admits a bi-invariant Riemannian metric.

Since every two points are connected by a geodesic, every element of a compact group is an exponent of an element of the Lie algebra, and every finite cyclic subgroup belongs to a circle subgroup. If we normalize the metric such that the diameter of each circle subgroup is bounded from below by $1/2$, then the diameter of each finite subgroup is at least $1/3$, which is realized by $\mathbb{Z}/3\mathbb{Z}$. In particular, if we choose the bi-invariant metric on the group $SU(2) = S^3$ such that $\text{diam}(SU(2)) = 1/2$ (this is equivalent to each meridian circle being of length 1), the bound $1/3$ is realized.

An ε -representation of a group Γ is a map $q : \Gamma \rightarrow G$ such that $d(q(x)q(y), q(xy)) < \varepsilon$ for any $x, y \in \Gamma$. An ε -representation can be **δ -approximated by a representation** if there exists a representation $\rho : \Gamma \rightarrow G$ such that $d(\rho(x), q(x)) < \delta$ for all $x \in \Gamma$.

Theorem 5.3: Let M be a closed manifold of strictly negative sectional curvature, G a connected compact Lie group, and P a trivial principal G -bundle. For any connection ∇ in P , let $q_\nabla : \pi_1(M) \rightarrow G$ denote the corresponding HBG-quasimorphism (Subsection 3.5). Choose a left-invariant metric on G such that the diameter of any closed subgroup is at least $1/3$. Then for each $\varepsilon > 0$, there exists a connection ∇ such that q_∇ is an ε -representation which cannot be $1/3$ -approximated by a representation.

Proof. Step 1: Let $a_1, \dots, a_n$ be the generators of $\pi_1(M)$. Using Corollary 4.10, we find $b \in \pi_1(M)$ such that $b \not\sim_{\text{vce}} a_i$. Choose an open set U_b which intersects F_b and does not intersect $F_{a_1}, F_{a_2}, \dots, \nu(p, p_i)$, $i = 1, \dots, n$, where $\nu(p, p_i)$ are geodesics connecting the marked point p with F_{a_i} . Let p_b be the chosen fixed point on F_b , and ∇_0 be the trivial connection on a trivial G -bundle P . We are going to construct a connection $\nabla_0 + \theta$ which is trivial outside of U_b , by choosing an appropriate 1-form $\theta \in \Lambda^1(M) \otimes \mathfrak{g}$, where $\mathfrak{g} = \text{Lie}(G)$. Consider the path $\gamma = \nu(p, p_b)F_b\nu(p, p_b)^{-1}$ as a map from $[0, 1]$ to M . Then the holonomy g of $\nabla + \theta$ along γ satisfies $g_t(t_0) = \int_0^{t_0} ((g_t)^{-1})^* \dot{g}_t dt$,

where $((g_t)^{-1})^* \dot{g}_t = \theta|_t$.

For any non-trivial $g = e^u \in G$, we can choose a 1-form θ on M such that $\gamma^*(\theta) = f(t)u dt$, for some function $f : [0, 1] \rightarrow \mathbb{R}$ with compact support. Choosing f such that $g = \int_0^1 f(t)u dt$, we obtain a 1-form θ with values in the Lie algebra of G such that the holonomy of $\nabla_0 + \theta$ along γ is equal to g . Moreover, replacing θ with $\frac{1}{m}\theta$, we obtain a connection with holonomy $e^{\frac{1}{m}u} = g^{1/m}$.

For m sufficiently large, this would give an ε -representation q_∇ such that $q_\nabla(a_i) = 1_G$, and $q_\nabla(b)^m = g$.

Step 2: It remains to show that the ε -representation q_∇ cannot be $1/3$ -approximated by a representation ρ . By contradiction, assume that q_∇ is $1/3$ -approximated by a representation $\rho : \pi_1(M) \rightarrow G$. Since $d(\rho(a_i^n), q_\nabla(a_i)^n) < 1/3$, the closure of a subgroup of G generated by $\rho(a_i)$ has diameter less than $1/3$. Since the diameter of non-trivial subgroups of G is $\geq 1/3$, this implies that $\rho(a_i) = 1_G$, and ρ is trivial. Therefore, $\rho(b) = 1_G$. However, for all $n \in \mathbb{Z}$, we have $d(\rho(b)^n, q_\nabla(b)^n) < 1/3$, because q_∇ is $1/3$ -approximated by ρ . Then the diameter of the subgroup of G generated by $g = q_\nabla(b)$ is less than $1/3$, which again implies that $g = 1_G$, leading to contradiction. ■

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References

- [BG] J. Barge, É. Ghys, *Surfaces et cohomologie bornée*, Invent. Math. 92 (1988), no. 3, 509-526. (Cited on pages 5, 6, and 18.)
- [BB] L. Bartholdi and O. Bogopolski, *On abstract commensurators of groups*, J. Group Theory, 13(6):903-922, 2010. (Cited on page 20.)
- [BFMSS] L. Battista, S. Francaviglia, M. Moraschini, F. Sarti, and A. Savini, *Bounded cohomology classes of exact forms*, Proc. Amer. Math. Soc., To appear. arXiv:2211.16125, 2022. (Cited on page 6.)
- [B] C. Bavard, *Longueur stable des commutateurs*, Enseign. Math. (2) 37 (1991), no. 1-2, 109-150. (Cited on page 5.)

- [BBF] Bestvina, M., Bromberg, K. and Fujiwara, K., *Bounded cohomology with coefficients in uniformly convex Banach spaces*, 2016, Commentarii Mathematici Helvetici, 91(2), pp.203-218. (Cited on page 4.)
- [BF] M. Bestvina and K. Fujiwara *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol., 6, pp 69-89, 2002. (Cited on pages 25 and 26.)
- [BM] M. Brandenbursky, M. Marcinkowski, *Entropy and quasimorphisms*, Source: Journal of Modern Dynamics (2019), Vol. 15, p 143-163. (Cited on page 5.)
- [BV] Michael Brandenbursky, Misha Verbitsky, *Non-commutative Barge-Ghys quasimorphisms*, Int. Math. Res. Not. IMRN 2024, no. 15, 11135-11158. (Cited on pages 4, 10, 17, and 29.)
- [BH] M. R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999. (Cited on pages 19 and 20.)
- [Br] R. Brooks, *Some remarks on bounded cohomology*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pp. 53-63, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981. (Cited on pages 4, 5, and 25.)
- [BBI] D. Burago, Yu. Burago, S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001. (Cited on page 13.)
- [BM1] M. Burger and N. Monod, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. (JEMS) 1 (2) (1999), 199-235. (Not cited.)
- [BM2] Burger, M., and Monod, N., *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. 12 (2) (2002), 219-280. (Not cited.)
- [BOT] M. Burger, N. Ozawa, A. Thom, *On Ulam stability*, Israel J. Math. 193 (2013), no. 1, 109-129. (Cited on pages 8 and 9.)
- [C] D. Calegari, *scl*, MSJ Memoirs, 20. Mathematical Society of Japan, Tokyo, 2009. (Cited on pages 5, 6, 7, 18, 20, 22, 23, and 25.)
- [DFHM] Sami Douba, Francesco Fournier-Facio, Sam Hughes, Simon Machado *Quasihomomorphisms to real algebraic groups*, arXiv:2601.22411 (Cited on pages 4, 9, and 10.)
- [DGO] F. Dahmani, V. Guirardel, and D. Osin. *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*. Mem. Amer. Math. Soc., 245(1156):v+152, 2017. (Cited on pages 24, 25, and 26.)
- [dH] P. de la Harpe, *Spaces of closed subgroups of locally compact groups*, 2008, arXiv:0807.2030. (Cited on page 32.)

- [EF] D. B. A. Epstein, K. Fujiwara, *The second bounded cohomology of word hyperbolic groups*, Topology 36, (1997), 1275-1289. (Cited on page 26.)
- [EP] M. Entov, L. Polterovich, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. 2003, no. 30, 1635-1676. (Cited on pages 5 and 6.)
- [F] J. M. G. Fell, *A Hausdorff topology for the closed subsets of a locally compact non Hausdorff space*, Proc. Amer. Math. Soc. 13 (1962), 472-476. (Cited on page 32.)
- [FPS] R. Frigerio, M. B. Pozzetti, and A. Sisto., *Extending higher-dimensional quasi-cocycles*. J.Topol., 8(4):1123-1155, 2015. (Cited on page 25.)
- [Fri] R. Frigerio, *Bounded Cohomology of Discrete Groups*, Mathematical Surveys and Monographs 227, AMS, 2017. (Cited on page 13.)
- [FK] K. Fujiwara, M. Kapovich, *On quasihomomorphisms with noncommutative targets*, Geom. Funct. Anal. 26 (2016), no. 2, 478-519. (Cited on pages 2, 3, 4, 9, 10, and 26.)
- [GG] J.-M. Gambaudo, É. Ghys, *Commutators and diffeomorphisms of surfaces*. Ergodic Theory Dynam. Systems 24 (2004), no. 5, 1591-1617. (Cited on page 5.)
- [GLMR] L. Glebsky, A. Lubotzky, N. Monod, B. Rangarajan, *Asymptotic Cohomology and Uniform Stability for Lattices in Semisimple Groups*, arXiv:2301.00476. (Cited on page 9.)
- [G2] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. No. 56 (1982), 5-99 (1983). (Cited on pages 5, 13, and 14.)
- [HO] M. Hull, D. Osin. *Induced quasicocycles on groups with hyperbolically embedded subgroups*, Algebr. Geom. Topol., 13 (5), pp. 2635-2665, 2013. (Cited on page 25.)
- [HS] T. Hartnick, P. Schweitzer, *On quasioutomorphism groups of free groups and their transitivity properties*, Journal of Algebra, Volume 450 (2016), pp. 242-281, arXiv:1403.2786. (Cited on page 4.)
- [Kaz] D. Kazhdan, *On ε -representations*, Israel Journal of Mathematics, vol. 43, pages 315-323 (1982), (Cited on pages 3, 4, 8, and 32.)
- [Kl] W. P. A. Klingenberg, *Riemannian Geometry*, second edition (1995), de Gruyter. (Cited on page 15.)
- [KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Wiley-Interscience, New York (1969). (Cited on pages 11 and 12.)
- [Mar] D. Marasco, *Cup product in bounded cohomology of negatively curved manifolds*, Proc. Amer. Math. Soc., 151(6):2707-2715, 2023. (Cited on page 6.)

- [MO1] Users of MathOverflow,
<https://mathoverflow.net/questions/423909/holonomy-bounded-in-terms-of-area-and-the-curvature>
(Cited on page 16.)
- [MO2] Y. Cornuier, S. Nead, Mathoverflow question
<https://mathoverflow.net/questions/423909/holonomy-bounded-in-terms-of-area-and-the-curvature>
(Cited on page 20.)
- [M] I. Mineyev, *Straightening and bounded cohomology of hyperbolic groups*, Geom. Funct. Anal., 11(4):807-839, 2001. (Cited on page 16.)
- [MW] T. Murphy, F. Wilhelm, *Random manifolds have no totally geodesic submanifolds* Michigan Math. J. 68 (2019), no. 2, 323-335. (Cited on page 14.)
- [vN] J. von Neumann, *Beweis des Ergodensatzes und des H-Theorems in der neuen Mechanik*, Z. Physik, 57:30-70, 1929. (Cited on page 8.)
- [RW] H. Reckziegel, E. Wilhelmus, *How the Curvature Generates the Holonomy of a Connection in an Arbitrary Fibre Bundle*, Results in Mathematics volume 49, pages 339-359 (2006). (Cited on page 16.)
- [OV] L. Ornea, M. Verbitsky, *Principles of Locally Conformally Kähler Geometry*, arXiv:2208.07188. (Cited on pages 12 and 18.)
- [O] D. Osin, *Acylically hyperbolic groups*. Trans. Amer. Math. Soc., 368(2):851-888, 2016. (Cited on page 25.)
- [P] P. Petersen, *Riemannian Geometry*, Third Edition, Graduate Texts in Mathematics, Springer Verlag. (Cited on page 20.)
- [PR] L. Polterovich, D. Rosen, *Function theory on symplectic manifolds*, CRM Monograph Series, 34, AMS, Providence, RI, 2014. (Cited on page 6.)
- [R] P. Rolli, *Quasi-morphisms on free groups*, arXiv preprint arXiv:0911.4234, 2009. (Cited on page 4.)
- [Sh] E. Shelukhin, *The Action homomorphism, quasimorphisms and moment maps on the space of compatible almost complex structures*, Commentarii Mathematici Helvetici (2014), Volume 89, Issue 1, 2014, 69–123. (Cited on page 5.)
- [St] S. Sternberg *Lectures on differential geometry*, (1964) New York: Chelsea (1093). (Cited on page 11.)
- [Tao] T. Tao, *Hilbert’s fifth problem and related topics*, Graduate Studies in Mathematics, AMS, 2014. (Cited on page 32.)
- [Tu] A. M. Turing, *Finite approximations to Lie groups*, Ann. of Math. (2), 39(1):105-111, 1938. (Cited on page 8.)
- [U] S. M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8 Interscience Publishers, New York-London 1960. (Cited on page 7.)

[Y] D. Yang, *Holonomy equals curvature*, <https://cims.nyu.edu/~yangd/papers/holonomy.pdf> (Cited on page 16.)

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