Algebraic geometry: final exam

Rules: Every student receives from me a list of 4 exercises (chosen randomly), and has to solve 2 of them by January 17. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solution, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes). Maximal score is 2 out of 4 exercises, but every exercise you solve brings you closer. Feel free to google the solutions if you are able.

Exercise 1.1. Let $M \subset \mathbb{C}^n$ be an algebraic subvariety of $\mathbb{C}^n$. Prove that M has a smooth point.

Exercise 1.2. Let M be an irreducible algebraic subvariety in $\mathbb{C}^n$. Prove that the set of smooth points of M is connected.

Exercise 1.3. Let M be a complex variety, $x \in M$ a smooth point, and $\mathcal{O}_x$ the local ring obtained by inverting the set S of all functions which do not vanish in x. Prove that $\mathcal{O}_x$ is factorial.

Exercise 1.4. Let I be an ideal in a Noetherian ring. Prove that $\bigcap_k I^k = 0$.

Definition 1.1. A module M over a ring R is called **flat** if the functor $-\otimes M$ is exact.

Exercise 1.5. Prove that a finitely generated module over a Noetherian local ring is flat if and only if it is free.

Exercise 1.6. Let K_1, K_2 be finite extensions of $\mathbb{Q}$. Prove that the ring $K_1 \otimes_{\mathbb{Q}} K_2$ is a direct sum of fields.

Exercise 1.7. Find finite extensions K_1, K_2 of a field K such that the ring $K_1 \otimes_K K_2$ contains nilpotents.

Exercise 1.8. Let C be a smooth 1-dimensional algebraic variety, equipped with an action of a finite group G. Prove that C/G is also smooth.

Exercise 1.9. Let R_1, R_2 be rings over $\mathbb{C}$ such that $R_1 \otimes_{\mathbb{C}} R_2$ is finitely generated. Prove that R_1, R_2 are finitely generated, or find a counterexample.