Algebraic geometry

Lecture 2: category of affine varieties

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Algebraic sets in $\mathbb{C}^n$ (reminder)

REMARK: In most situations, you can replace your ground field $\mathbb{C}$ by any other field. However, there are cases when chosing $\mathbb{C}$ as a ground field simplifies the situation. Moreover, using $\mathbb{C}$ is essentially the only way to apply topological arguments which help us to develop the geometric intuition.

DEFINITION: A subset $Z \subset \mathbb{C}^n$ is called **an algebraic set** if it can be goven as a set of solutions of a system of polynomial equations $P_1(z_1, ..., z_n) = P_2(z_1, ..., z_n) = ... = P_k(z_1, ..., z_n) = 0$, where $P_i(z_1, ..., z_n) \in \mathbb{C}[z_1, ..., z_n]$ are polynomials.

DEFINITION: **Algebraic function** on an algebraic set $Z \subset \mathbb{C}^n$ is a restriction of a polynomial function to Z. An algebraic set with a ring of algebraic functions on it is called **an affine variety**.

DEFINITION: Two affine varieties A, A' are **isomorphic** if there exists a bijective polynomial map $A \longrightarrow A'$ such that its inverse is also polynomial.

Maximal ideals (reminder)

REMARK: All rings are assumed to be commutative and with unit.

DEFINITION: An ideal I in a ring R is a subset $I \subsetneq R$ closed under addition, and such that for all $a \in I, f \in R$, the product fa sits in I. The quotient group R/I is equipped with a structure of a ring, called **the quotient ring**.

DEFINITION: A maximal ideal is an ideal $I \subset R$ such that for any other ideal $I' \supset I$, one has $I = I'$.

EXERCISE: Prove that **an ideal $I \subset R$ is maximal if and only if R/I is a field.**

THEOREM: Let $I \subset R$ be an ideal in a ring. **Then I is contained in a maximal ideal.**

Proof: One applies the Zorn lemma to the set of all ideals, partially ordered by inclusion. ■

Hilbert's Nullstellensatz (reminder)

EXAMPLE: Let A be an affine variety, $\mathcal{O}_A$ the ring of polynomial functions on A, $a \in A$ a point, and $I_a \subset \mathcal{O}_A$ an ideal of all functions vanishing in a. **Then I_a is a maximal ideal.**

DEFINITION: The ideal I_a is called **the (maximal) ideal of the point** $a \in A$.

THEOREM: (Hilbert's Nullstellensatz)
Let $A \subset \mathbb{C}^n$ be an affine variety, and $\mathcal{O}_A$ the ring of polynomial functions on A. **Then every maximal ideal in A is an ideal of a point $a \in A$: $I = I_a$.**

Categories

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: A class $\mathcal{Ob}(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{Ob}(\mathcal{C})$, one has a set $\mathcal{Mor}(X,Y)$ of **morphisms from X to Y**.

Composition of morphisms: For each $\varphi \in \mathcal{Mor}(X,Y), \psi \in \mathcal{Mor}(Y,Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{Mor}(X,Z)$

Identity morphism: For each $A \in \mathcal{Ob}(\mathcal{C})$ there exists a morphism $\operatorname{Id}_A \in \mathcal{Mor}(A,A)$.

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{Mor}(X,Y)$, one has $\operatorname{Id}_x \circ \varphi = \varphi = \varphi \circ \operatorname{Id}_Y$

Categories (2)

DEFINITION: Let $X, Y \in \mathcal{Ob}(\mathcal{C})$ – objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{Mor}(X, Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{Mor}(Y, X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:

Category of sets: its morphisms are arbitrary maps.

Category of vector spaces: its morphisms are linear maps.

Categories of rings, groups, fields: morphisms are homomorphisms.

Category of topological spaces: morphisms are continuous maps.

Category of smooth manifolds: morphisms are smooth maps.

Functors

DEFINITION: Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories. A **covariant functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the following set of data.

1. **A map** $F: \mathcal{Ob}(\mathcal{C}_1) \longrightarrow \mathcal{Ob}(\mathcal{C}_2)$.

2. **A map** $F: \mathcal{Mor}(X, Y) \longrightarrow \mathcal{Mor}(F(X), F(Y))$ **defined for any pair of objects** $X, Y \in \mathcal{Ob}(\mathcal{C}_1)$**.**

These data define a functor if they are **compatible with compositions,** that is, satisfy $F(\varphi) \circ F(\psi) = F(\varphi \circ \psi)$ for any $\varphi \in \mathcal{Mor}(X, Y)$ and $\psi \in \mathcal{Mor}(Y, Z)$, and **map identity morphism to identity** morphism.

Small categories

REMARK: This way, one could speak of **category of all categories**, with categories as objects and functors as morphisms.

A caution To avoid set-theoretic complications, Grothendieck added another axiom to set theory, "universum axiom", postulating existence of "universum", a very big set, and worked with "small categories" – categories where the set of all objects and sets of morphisms belong to the universum. In this sense, "category of all categories" is not a "small category", because the set of its object (being comparable to the set of all subsets of the universum) is too big to fit in the universum.

In practice, mathematicians say "category" when they mean "small category", tacitly assuming that any given category is "small". This is why not many people call "category of all categories" a category: nobody wants to deal with set-theoretic complications.

Example of functors

A "natural operation" on mathematical objects is usually a functor. Examples:

1. A map $X \longrightarrow 2^X$ from the set X to the set of all subsets of X is a functor from the category $\mathcal{S}ets$ of sets to itself.

2. A map $M \longrightarrow M^2$ mapping a topological space to its product with itself is a functor on topological spaces.

3. A map $V \longrightarrow V \oplus V$ is a functor on vector spaces; same for a map $V \longrightarrow V \otimes V$ or $V \longrightarrow (V \oplus V) \otimes V$.

4. **Identity functor** from any category to itself.

5. A map from topological spaces to $\mathcal{S}ets$, putting a topological space to the set of its connected components.

EXERCISE: Prove that it is a functor.

Contravariant functors

DEFINITION: Let $\mathcal{C}$ be a category. Define the **opposite category** $\mathcal{C}^{op}$ with the same set of objects, and $\mathcal{M}or_{\mathcal{C}^{op}}(A,B) = \mathcal{M}or_{\mathcal{C}}(B,A)$. The composition $\varphi \circ \psi$ in $\mathcal{C}$ gives the composition $\psi^{op} \circ \varphi^{op}$ in $\mathcal{C}^{op}$.

DEFINITION: A **contravariant functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the usual ("covariant") functor from $\mathcal{C}_1$ to $\mathcal{C}_2^{op}$.

EXAMPLE: A map from the category of topological spaces to category of rings mapping a space to a ring of continuous functions on it gives a contravariant functor.

EXAMPLE: Let $X \in \mathcal{O}b(\mathcal{C})$ be an object of $\mathcal{C}$. A map $Y \longrightarrow \mathcal{M}or(X,Y)$ defines a covariant functor from $\mathcal{C}$ to the category $\mathcal{S}ets$ of sets. A map $Y \longrightarrow \mathcal{M}or(Y,X)$ defines a contravariant functor from $\mathcal{C}$ to $\mathcal{S}ets$. Such functors to $\mathcal{S}ets$ are called **representable**.

Equivalence of functors

DEFINITION: Let $X, Y \in \mathcal{O}b(\mathcal{C})$ be objects of a category $\mathcal{C}$. A mprphism $\varphi \in \mathcal{M}or(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}or(Y,X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case X and Y are called **isomorphic**.

DEFINITION: Two functors $F, G: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ are called **equivalent** if for any $X \in \mathcal{O}b(\mathcal{C}_1)$ we are given an isomorphism $\Psi_X: F(X) \longrightarrow G(X)$, in such a way that for any $\varphi \in \mathcal{M}or(X,Y)$, one has $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$.

REMARK: Such commutation relations are usually expressed by **commutative diagrams**. For example, the condition $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$ is expressed by a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(\varphi)"] \arrow[d, "\Psi_X"'] & F(Y) \arrow[d, "\Psi_Y"] \\ G(X) \arrow[r, "G(\varphi)"] & G(Y) \end{tikzcd}$$

Equivalence of categories

DEFINITION: A functor $F: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is called **equivalence of categories** if there exists a functor $G: \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that the compositions $G \circ F$ and $G \circ F$ are equivaleent to the identity functors $\operatorname{Id}_{\mathcal{C}_1}$, $\operatorname{Id}_{\mathcal{C}_2}$.

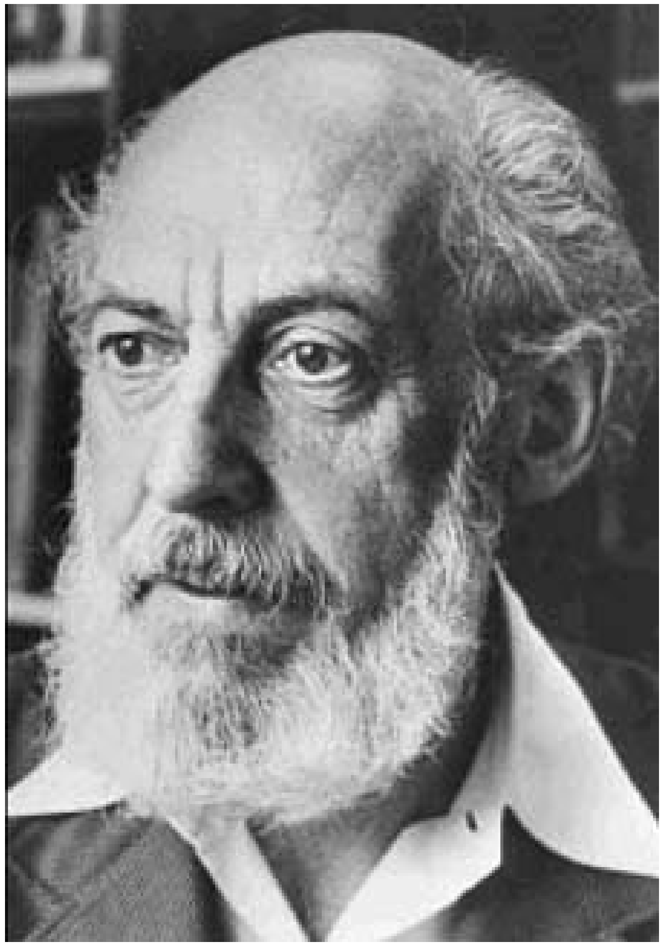
REMARK: It is possible to show that this is equivalent to the following conditions: **F defines a bijection on the set of isomorphism classes of objects of $\mathcal{C}_1$ and $\mathcal{C}_2$, and a bijection**

$$\mathcal{M}or(X,Y) \longrightarrow \mathcal{M}or(F(X),F(Y)).$$

for each $X, Y \in \mathcal{O}b(\mathcal{C}_1)$.

REMARK: From the point of view of category theory, **equivalent categories are two instances of the same category** (even if the cardinality of corresponding sets of objects is different).

Saunders Mac Lane
(1909-2005)



Samuel Eilenberg
(1913-1998)

Alexander Grothendieck
(. 28 1928)

Category of affine varieties and category of finitely generated rings

DEFINITION: Category of affine varieties over $\mathbb{C}$: its objects are algebraic subsets in $\mathbb{C}^n$, morphisms – polynomial maps.

DEFINITION: Finitely generated ring over $\mathbb{C}$ is a quotient of $\mathbb{C}[t_1, ..., t_n]$ by an ideal.

DEFINITION: Let R be a ring. An element $x \in R$ is called **nilpotent** if $x^n = 0$ for some $n \in \mathbb{Z}^{>0}$.

Theorem 1: Let $\mathcal{C}_R$ be a category of finitely generated rings over $\mathbb{C}$ without non-zero nilpotents and $\mathcal{Aff}$ – category of affine varieties. Consider the functor $\Phi: \mathcal{Aff} \longrightarrow \mathcal{C}_R^{op}$ mapping an algebraic variety X to the ring of polynomial functions on X. **Then Φ is an equivalence of categories.**

Proof: Will take the rest of this lecture.

Strong Nullstellensatz

DEFINITION: Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal. Denote the **set of common zeros** for I by $V(I)$, with

$$V(I) = \{(z_1, ..., z_n) \in \mathbb{C}^n \ \mid \ f(z_1, ..., z_n) = 0 \forall f \in I\}.$$

For $Z \subset \mathbb{C}^n$ an algebraic subset, denote by $\operatorname{Ann}(A)$ the set of all polynomials $P(t_1, ..., t_n)$ vanishing in Z.

THEOREM: (strong Nullstellensatz). For any ideal $I \subset \mathbb{C}[t_1, ..., t_n]$ such that $\mathbb{C}[t_1, ..., t_n]/I$ has no nilpotents, **one has** $\operatorname{Ann}(V(I)) = I$**.**

Proof: Later in this lecture.

REMARK: "Weak Nulstellensatz" claims that $V(I)$ is never empty for any non-trivial ideal I; "Weak Nulstellensatz" claims that I is determined by $V(I)$ when R/I has no nilpotents.

Strong Nullstellensatz and equivalence of categories

THEOREM: (strong Nullstellensatz). For any ideal $I \subset \mathbb{C}[t_1, ..., t_n]$ such that $\mathbb{C}[t_1, ..., t_n]/I$ has no nilpotents, **one has** $\operatorname{Ann}(V(I)) = I$**.**

Now we deduce Theorem 1 from Strong Nullstellensatz. This would require us to construct a functor $\Psi : \mathcal{C}_R^{op} \longrightarrow \mathcal{A}\!f\!f$. Since any object $R \in \mathcal{O}b(\mathcal{C}_R)$ is given as $R = \mathbb{C}[t_1, ..., t_n]/I$, we define Ψ as $\Psi(R) = V(I)$; the functor $\Phi : \mathcal{A}\!f\!f \longrightarrow \mathcal{C}_R^{op}$ was defined as $Z \longrightarrow \operatorname{Ann}(Z)$.

Strong Nullstellensatz gives $\operatorname{Ann}(V(I)) = I$, hence $\Phi(\Psi(R)) = R$ **for any finitely generated ring.** It remains to prove $V(\operatorname{Ann}(Z)) = Z$.

Clearly, $V(\operatorname{Ann}(Z)) \supset Z$: any point $z \in Z$ belongs to the set of common zeros of $\operatorname{Ann}(Z)$. On the other hand, Z is a set of common zeros of a system $\mathcal{P}$ of polynomial equations, giving $Z = V(\mathcal{P}) \supset V(\operatorname{Ann}(Z))$.

Localization

DEFINITION: Localization of a ring R with respect to $F \in R$ is a ring $R[F^{-1}]$, which is formally generated by the elements of form a/F^n and relations $a/F^n \cdot b/F^m = ab/F^{n+m}$, $a/F^n + b/F^m = \frac{aF^m + bF^n}{F^{n+m}}$, and $aF^k/F^{k+n} = a/F^n$.

EXAMPLE: $\mathbb{Z}[2^{-1}]$, the ring of rational numbers with denominators 2^k.

EXAMPLE: $\mathbb{C}[t, t^{-1}]$, the ring of Laurent polynomials.

EXERCISE: Let R be a finitely generated ring over a field k. **Prove that $R[F^{-1}]$ is a finitely generated ring over k.**

CLAIM 1: Suppose that $R[F^{-1}] = 0$, where $F \in R$. **Then F is nilpotent.**

Proof. Step 1: $R(F) = R[t]/(tF - 1)$. **Therefore, $1 = 0$ implies $1 = (Ft - 1)P$, for some $P \in R[t]$.**

Step 2: Let $P(t) = \sum a_i t^i$, where $a_i \in R$. **Then $1 = (Ft - 1)P$ implies $a_i = a_{i-1}F$ for all $i > 0$, and $a_0 = 1$.**

Step 3: This gives $P = \sum F^i t^i$, and $F^{n+1} = 0$. ■

Spectrum and localization

DEFINITION: Spectrum of a ring R is the set $\operatorname{Spec} R$ if its prime ideals.

EXERCISE: Let $R \xrightarrow{\varphi} R_1$ be a ring homomorphism. Prove that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal, for any $\mathfrak{p} \in \operatorname{Spec} R_1$.

PROPOSITION: In other words, any morphism $R \longrightarrow R_1$ **gives an injective map of spectra** $\operatorname{Spec} R[f^{-1}] \hookrightarrow \operatorname{Spec} R$.

Proof: Suppose that $\mathfrak{p}_f, \mathfrak{q}_f \in \operatorname{Spec} R(f)$, and $\mathfrak{p} = \mathfrak{q}$ are their images in $\operatorname{Spec} R$. Then for each $p \in \mathfrak{p}_f$, we have $f^N p \in \mathfrak{q} \subset \mathfrak{q}_f$; since $\mathfrak{q}$ is prime, this implies that $p \in \mathfrak{q}$. ■

DEFINITION: Nilradical of a ring R is the set $\operatorname{Nil}(R)$ of all nilpotent elements of R.

THEOREM: Interesection P of all prime ideals of R is equal to $\operatorname{Nil}(R)$.

Proof: Clearly, $P \supset \operatorname{Nil}(R)$. Assume that, conversely, $x \notin \operatorname{Nil}(R)$. Then $R[x^{-1}] \neq 0$, **hence $R[x^{-1}]$ contains a prime ideal** (the maximal one), and its image in $\operatorname{Spec} R$ does not contaim x. ■

Rabinowitz trick

DEFINITION: Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal. Recall that the set of common zeros of I is denoted by $V(I)$ **("vanishing set", "null-set", "zero set"),** and the set of all polynomials vanishing in $Z \subset \mathbb{C}^n$ is denoted $\operatorname{Ann}(Z)$ **("annihilator").**

Theorem 1: Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal, and f a polynomial function, vanishing on $V(I)$. **Then $f^N \in I$ for some $N \in \mathbb{Z}^{>0}$.**

Proof. Step 1: Consider an ideal $I_1 \subset \mathbb{C}[t_1, ..., t_{n+1}]$ generated by $I \subset \mathbb{C}[t_1, ..., t_n]$ and $ft_{n+1} - 1$. **Since $ft_{n+1} - 1, I$ have no common zeros, I_1 contains 1** by (weak) Nulstellensatz.

Step 2: Let $R := \mathbb{C}[t_1, ..., t_n]/I$. Consider the map $\zeta : \mathbb{C}[t_1, ..., t_{n+1}] \longrightarrow R[f^{-1}]$ which is identity on $t_1, ..., t_n$ and mapping t_{n+1} to f^{-1}. Since $\zeta(I_1) = 0$, and $1 \in I_1$, one has $1 = 0$ in $R[f^{-1}]$, giving $R[f^{-1}] = 0$. **By Claim 1, f is nilpotent.** ■

COROLLARY: (Strong Nullstellensatz)
Suppose that $R := \mathbb{C}[t_1, ..., t_n]/I$ is a ring without nilpotents. **Then $I = \operatorname{Ann}(V_I))$.**

Proof: If $a \in \operatorname{Ann}(V_I))$, then $a^n \in I$ by Theorem 1. ■