Algebraic geometry

Lecture 3: irreducible varietiees and Noetherian rings

Misha Verbitsky

Université Libre de Bruxelles

October 13, 2015

Algebraic sets in $\mathbb{C}^n$ (reminder)

REMARK: In most situations, you can replace your ground field $\mathbb{C}$ by any other field. However, there are cases when chosing $\mathbb{C}$ as a ground field simplifies the situation. Moreover, using $\mathbb{C}$ is essentially the only way to apply topological arguments which help us to develop the geometric intuition.

DEFINITION: A subset $Z \subset \mathbb{C}^n$ is called **an algebraic set** if it can be goven as a set of solutions of a system of polynomial equations $P_1(z_1, ..., z_n) = P_2(z_1, ..., z_n) = ... = P_k(z_1, ..., z_n) = 0$, where $P_i(z_1, ..., z_n) \in \mathbb{C}[z_1, ..., z_n]$ are polynomials.

DEFINITION: **Algebraic function** on an algebraic set $Z \subset \mathbb{C}^n$ is a restriction of a polynomial function to Z. An algebraic set with a ring of algebraic functions on it is called **an affine variety**.

DEFINITION: Two affine varieties A, A' are **isomorphic** if there exists a bijective polynomial map $A \longrightarrow A'$ such that its inverse is also polynomial.

Maximal ideals (reminder)

REMARK: All rings are assumed to be commutative and with unit.

DEFINITION: An ideal I in a ring R is a subset $I \subsetneq R$ closed under addition, and such that for all $a \in I, f \in R$, the product fa sits in I. The quotient group R/I is equipped with a structure of a ring, called **the quotient ring**.

DEFINITION: A maximal ideal is an ideal $I \subset R$ such that for any other ideal $I' \supset I$, one has $I = I'$.

EXERCISE: Prove that **an ideal $I \subset R$ is maximal if and only if R/I is a field.**

THEOREM: Let $I \subset R$ be an ideal in a ring. **Then I is contained in a maximal ideal.**

Proof: One applies the Zorn lemma to the set of all ideals, partially ordered by inclusion. ■

Hilbert's Nullstellensatz (reminder)

EXAMPLE: Let A be an affine variety, $\mathcal{O}_A$ the ring of polynomial functions on A, $a \in A$ a point, and $I_a \subset \mathcal{O}_A$ an ideal of all functions vanishing in a. **Then I_a is a maximal ideal.**

DEFINITION: The ideal I_a is called **the (maximal) ideal of the point** $a \in A$.

THEOREM: (Hilbert's Nullstellensatz)
Let $A \subset \mathbb{C}^n$ be an affine variety, and $\mathcal{O}_A$ the ring of polynomial functions on A. **Then every maximal ideal in A is an ideal of a point $a \in A$: $I = I_a$.**

DEFINITION: Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal. Denote the **set of common zeros** for I by $V(I)$, with

$$V(I) = \{(z_1, ..., z_n) \in \mathbb{C}^n \ \mid \ f(z_1, ..., z_n) = 0 \forall f \in I\}.$$

For $Z \subset \mathbb{C}^n$ an algebraic subset, denote by $\operatorname{Ann}(A)$ the set of all polynomials $P(t_1, ..., t_n)$ vanishing in Z.

THEOREM: (strong Nullstellensatz). For any ideal $I \subset \mathbb{C}[t_1, ..., t_n]$ such that $\mathbb{C}[t_1, ..., t_n]/I$ has no nilpotents, **one has $\operatorname{Ann}(V(I)) = I$.**

Categorical equivalence (reminder)

DEFINITION: Category of affine varieties over $\mathbb{C}$: its objects are algebraic subsets in $\mathbb{C}^n$, morphisms – polynomial maps.

DEFINITION: Finitely generated ring over $\mathbb{C}$ is a quotient of $\mathbb{C}[t_1, ..., t_n]$ by an ideal.

DEFINITION: Let R be a ring. An element $x \in R$ is called **nilpotent** if $x^n = 0$ for some $n \in \mathbb{Z}^{>0}$. A ring which has no nilpotents is called **reduced**, and an ideal $I \subset R$ such that R/I has no nilpotents is called **a radical ideal**.

THEOREM: Let $\mathcal{C}_R$ be a category of finitely generated rings over $\mathbb{C}$ without non-zero nilpotents and $\mathcal{A}\!f\!f$ – category of affine varieties. Consider the functor $\Phi: \mathcal{A}\!f\!f \longrightarrow \mathcal{C}_R^{op}$ mapping an algebraic variety X to the ring $\mathcal{O}_X$ of polynomial functions on X. **Then Φ is an equivalence of categories.**

REMARK: Nulstellensatz implies that **points of X are in bijective correspondence with maximal ideals of $\mathcal{O}_X$. Prove it!**

Smooth points

DEFINITION: Let $A \subset \mathbb{C}^n$ is an algebraic subset. A point $a \in A$ is called **smooth**, or **smooth in a variety of dimension K** if there exists a neighbourhood U of $a \in \mathbb{C}^n$ such that $A \cap U$ is a smooth $2k$-dimensional real submanifold. A point is called **singular** if such diffeomorphism does not exist. A variety is called **smooth** if it has no singularities, and **singular** otherwise.

PROPOSITION: For any algebraic variety A and any smooth point $a \in A$, a diffeomorphism between a neighbourhood of a and an open ball **can be chosen polynomial**.

Proof. Step 1: Inverse function theorem. Let $a \in M$ be a point on a smooth k-dimensional manifold and $f_1, ..., f_k$ functions on M such that their differentials $df_1, ..., df_k$ are linearly independent in a. Then $f_1, ..., f_k$ **define a coordinate system in a neighbourhood of a, giving a diffeomorphism of this neighbourhood to an open ball.**

Step 2: If $a \in A \subset \mathbb{C}^n$ is a smooth point of a k-dimensional embedded manifold, **there exists k complex linear functions on $\mathbb{C}^n$ which are linearly independent on $T_a A$.**

Step 3: These function define **diffeomorphism from a neighbourhood of A to an open subset of $\mathbb{C}^k$.** ■

Maximal ideal of a smooth point

REMARK: The set of smooth points of A is open.

CLAIM: Let $\mathfrak{m}_x$ be a maximal ideal of a smooth point of a k-dimensional manifold M. **Then $\dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2 = k$.**

Proof: Consider a map $d_x : \mathfrak{m}_x \longrightarrow T_x^*M$ mapping a function f to $df|_x$. Clearly, d_x is surjective, and satisfies $\ker d_x = \mathfrak{m}_x^2$ **(prove it!)** ■

CLAIM: A manifold $A \subset \mathbb{C}^2$ given by equation $xy = 0$ **is not smooth in $a := (0, 0)$.**

Proof. Step 1: $\mathfrak{m}_a/\mathfrak{m}_a^2$ is the quotient of the space of all polynomials, vanishing in a, that is, degree $\geqslant 1$, by all polynomials of degree $\geqslant 2$, hence it is 2-dimensional.

Step 2: Therefore, if a is smooth point of A, A is 2-dimensional in a neighbourhood of $(0, 0)$. **However, outside if a, A is a line, hence 1-dimensional:** contradiction. ■

Hard to prove, but intiutively obvious observations

EXERCISE: Prove that **the set of smooth points of an affine variety is algebraic**.

Really hard exercise: Prove that **any affine variety over $\mathbb{C}$ contains a smooth point.**

EXERCISE: Using these two exercises, **prove that the set of smooth points of A is dense in A.**

Irreducible varietiees

DEFINITION: A affine manifold A is called **reducible** if it can be expressed as a union $A = A_1 \cup A_2$ of affine varieties, such that $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$. If such a decomposition is impossible, A is called **irreducible**.

CLAIM: An affine variety A is **irreducible** if and only if its ring of polynomial functions $\mathcal{O}_A$ **has no zero divizors.**

Proof: If $A = A_1 \cup A_2$ is a decomposition of A into a non-trivial union of subvarieties, choose a non-zero function $f \in \mathcal{O}_A$ vanishing at A_1 and g vanishing at A_2. The product of these non-zero functions vanishes in $A = A_1 \cup A_2$, **hence $fg = 0$ in $\mathcal{O}_A$.** Conversely, **if $fg = 0$, we decompose $A = V_f \cup V_g$.** ■

Irreducibility for smooth varieties

EXERCISE: Let M be an algebraic variety which is smooth and connected. **Prove that it is irreducible.**

COROLLARY: Let A be an affine manifold such that its set A_0 of smooth points is dense in A and connected. **Then A is irreducible.**

Proof: If f and g are non-zero function such that $fg = 0$, the ring of polynomial functions on A_0 contains zero divizors. However, **on a smooth, connected complex manifold the ring of polynomial functions has no zero divisors by analytic continuity principle.** ■

EXERCISE: Let $X \longrightarrow Y$ be a morphism of affine manifols, where X is irreducible, and its image in Y is dense. **Prove that Y is also irreducible.**

Noetherian rings and irreducible components

DEFINITION: A ring is called **Noetherian** if any increasing chain of ideals stabilizes: for any chain $I_1 \subset I_2 \subset I_3 \subset ...$ one has $I_n = I_{n+1} = I_{n+2} = ...$

DEFINITION: An irreducible component of an algebraic set A is an irreducible algebraic subset $A' \subset A$ such that $A = A' \cup A''$, and $A' \not\subset A''$.

Remark 1: Let $A_1 \supset A_2 \supset ... \supset A_n \supset ...$ be a decreasing chain of algebraic subsets in an algebraic variety. **Then the corresponding ideals form an increasing chain of ideals:** $\operatorname{Ann}(A_1) \subset \operatorname{Ann}(A_2) \subset \operatorname{Ann}(A_3) \subset ...$

THEOREM: Let A be an affine variety, and $\mathcal{O}_A$ its ring of polynomial functions. Assume that $\mathcal{O}_A$ is Noetherian. Then **A is a union of its irreducible components, which are finitely many.**

Proof: See the next slide. ∎

Remark 2: From the noetherianity and Remark 1 it follows that A **cannot contain a strictly decreasing infinite chain of algebraic subvarieties.**

Noetherian rings and irreducible components (2)

THEOREM: Let A be an affine variety, and $\mathcal{O}_A$ its ring of polynomial functions. Assume that $\mathcal{O}_A$ is Noetherian. Then **A is a union of its irreducible components, which are finitely many.**

Proof. Step 1: Each point $a \in A$ belongs to a certain irreducible component. Indeed, suppose that such a component does not exist. Then for each decomposition $A = A_1 \cup A_2$ of A onto algebraic sets, the set A_i containing a can be split non-trivially onto a union of algebraic sets, the component containing a can also be split, and so on, *ad infinitum*. This gives a strictly decreasing infinity sequence, a contradiction (Remark 2).

Step 2: We proved existence of an irreducible decomposition, and **it remains only to show that number of irreducible components of A is finite.** Let $A = \bigcup A_i$ be an irreducible decomposition. Then each A_i is not contained in the union of the rest of A_i.

Step 3: Let **algebraic closure** of a set $X \subset \mathbb{C}^n$ be the intersection of all algebraic subsets containing X. Clearly, it is algebraic **(prove it!)** Since $A = A_i \cup \bigcup_{j \neq i} A_j$, the algebraic closure B_i of $A \backslash A_i$ does not contain A_i. and **the sequence $B_1 \supset B_1 \cap B_2 \supset B_1 \cap B_2 \cap B_3 \subset ...$ decreases strictly, unless there are only finitely many irreducible components.** Applying Remark 2 again, we obtain that the number of B_i is finite. ■

Noetherian rings

DEFINITION: A finitely generated ring is a quotient of a polynomial ring.

THEOREM: (Hilbert's Basis Theorem)
Any finitely generated ring over a field is Noetherian.

Proof: Later in this lecture.

COROLLARY: For any affine manifold, **its ring of functions is Noetherian,** hence the the irreducible decomposition exists and is finite.

REMARK: It suffices to prove Hilbert's Basis Theorem for the ring of polynomials. Indeed, any finitely generaed ring is a quotient of the polynomial ring, but **the set of ideals of the quotient ring A/I is injectively mapped to the set of ideals of R.**

REMARK: Therefore, **Hilbert's Basis Theorem would follow if we prove that $R[t]$ is Noetherian for any Noetherian ring R.**

EXERCISE: Find an example of a ring which is not Noetherian.

Finitely generated ideals

DEFINITION: Finitely generated ideal in a ring is an ideal $\langle a_1, ..., a_n\rangle$ of sums $\sum b_i a_i$, where $\{a_i\}$ is a fixed finite set of elements of R, called **generators** of R.

LEMMA: Let $I \subset R$ be a finitely generated ideal, and $I_0 \subset I_1 \subset I_2 \subset ...$ an increasing chain of ideals, such that $\bigcup_n I_n = I$. **Then this chain stabilises**.

Proof: Let $I = \langle a_1, ..., a_n\rangle$, and I_N be an ideal in the chain $I_0 \subset I_1 \subset I_2 \subset ...$ which contains all a_i. Then $I_N = I$. ■

CLAIM: A ring R is Noetherian if and only if all its ideals are finitely generated.

Proof: For any chain of ideals $I_0 \subset I_1 \subset I_2 \subset ...$, **finite generatedness of $I = \bigcup I_i$ guarantees stabilization of this chain,** as follows from Lemma above.

Conversely, if R is Noetherian, and I any ideal, take $I_0 = 0$ and let $I_k \subset I$ be obtained by adding to I_{k-1} an element of I not containing in I_{k-1}. **Since the chain $\{I_k\}$ stabilizes, I is finitely generated.** ■

Noetherian modules

DEFINITION: A module over a ring R is a vector space M equipped with an algebra homomorphism $R \longrightarrow \operatorname{End}(M)$.

EXAMPLE: A subspace $I \subset R$ in a ring **is an ideal if and only if I is an R-submodule of R,** considered as an R-module.

DEFINITION: A module M over R is called **Noetherian** if any increasing chain of submodules of M stabilizes.

REMARK: Any submodules and quotient modules of a Noetherian R-module are again Noetherian.

Finitely generated R-modules

DEFINITION: An R-module is called **finitely generated** if it is a quotient of **a free module** R^n by its submodule.

EXERCISE: Show that **a module M is Noetherian iff any $M' \subset M$ is finitely generated.** Use this to prove that **direct sums of Noetherian modules are Noetherian.**

LEMMA: **A ring R is Noetherian if and only if it is Noetherian as an R-module.**

Proof: Ideals in R is the same as R-submodules of R, stabilization of a chain of R-submodules in R is literally the same as stabilization of a chain of ideals in R. ■

REMARK: Let M be a module over $R[t]$ which is Noetherian as an R-module, **Then it is Noetherian as $R[t]$-module.** ■

COROLLARY: If R is Noetherian, then $R[t]/(t^N) = R^N$ is a Noetherian R-module. Therefore, **the ring $R[t]/(t^N)$ is Noetherian.** ■

Proof of Hilbert's basis theorem

PROPOSITION: Let R be a Noetherian ring. **Then the polynomial ring $R[t]$ is also Noetherian.**

Proof. Step 1: Let $I \subset R[t]$ be an ideal. We need to show that it is finitely generated. Consider the ideal $I_0 \subset R$ generated by all leading coefficients of all $P(t) \in I$. Since R is Noetherian, I_0 is finitely generated: $I_0 = \langle a_1, ..., a_n \rangle$, where all a_i are leading coefficients of $P_i(t) \in I$.

Step 2: Let N be the maximum of all degrees of P_i. For each $Q(t) \in I$ with the leading coefficient $\sum a_i b_i$ **there exists a polynomial $P_Q(t)$ of degree no bigger than N with the same leading coefficient: $P_Q(T) = \sum_i P_i(t) b_i t^{N - \deg P_i}$.**

Step 3: Let $\tilde{Q}(t)$ be the remainder of the long division of $Q(t) \in I$ by $P_Q(y)$. Then $\tilde{Q}(t) = Q(t) \mod \langle P_1(t), ..., P_n(t) \rangle$, and $\deg \tilde{Q}(t) < N$.

Step 4: We have constructed an R-module embedding

$$M := I / \langle P_1(t), ..., P_n(t) \rangle \longrightarrow R[t]/(t^N).$$

Since M is a submodule of $R[t]/(t^N)$, it is a Noetherian module, as shown above, hence finitely generated. Pick a set of polynomials $Q_1(t), ..., Q_m(t) \in I$, generating M. **Then $\{Q_i(t), P_i(t)\}$ generate I.** ■