Algebraic geometry

Lecture 7: fibered product

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Tensor product (reminder)

DEFINITION: Let R be a ring, and M, M' modules over R. We denote by $M \otimes_R M'$ an R-module generated by symbols $m \otimes m'$, $m \in M, m' \in M'$, modulo relations
$r(m \otimes m') = (rm) \otimes m' = m \otimes (rm')$,
$(m + m_1) \otimes m' = m \otimes m' + m_1 \otimes m'$,
$m \otimes (m' + m'_1) = m \otimes m' + m \otimes m'_1$. Such an R-module is called **the tensor product of M and M' over R.**

REMARK: Suppose that M is generated over R by a set $\{m_i \in M\}$, and M' generated by $\{m'_j \in M'\}$. **Then $M \otimes_R M'$ is generated by $\{m_i \otimes m'_j\}$.**

EXERCISE: Find two non-zero R-modules A, B such that $A \otimes_R B = 0$ when

a. $R = \mathbb{Z}$.

b. $R = C^\infty M$ the ring of smooth functions on a manifold.

c. $R = \mathbb{C}[t]$ (polynomial ring).

Bilinear maps (reminder)

DEFINITION: Let M_1, M_2, M be modules over a ring R. **Bilinear map** $\mu(M_1, M_2) \xrightarrow{\varphi} M$ is a map satisfying $\varphi(rm, m') = \varphi(m, rm') = r\varphi(m, m')$, $\varphi(m + m_1, m') = \varphi(m, m') + \varphi(m_1, m')$, $\varphi(m, m' + m'_1) = \varphi(m, m') + \varphi(m, m'_1)$.

THEOREM: (Universal property of the tensor product)
For any bilinear map $B : M_1 \times M_2 \longrightarrow M$ **there exists a unique homomorphism $b : M_1 \otimes M_2 \longrightarrow M$, making the following diagram commutative:**

$$\begin{tikzcd}
M_1 \times M_2 \arrow[r, "B"] \arrow[dr, "\tau"] & M_1 \otimes M_2 \arrow[d, "b"] \\
& M
\end{tikzcd}$$

■

REMARK: If R is the field k, R-modules are vector spaces, and the previous theorem proves that $\mathrm{Bil}(M_1 \times M_2, k) = (M_1 \otimes M_2)^*$. For finite-dimensional M_i, it gives $M_1 \otimes M_2 = (M_1 \otimes M_2)^{**} = \mathrm{Bil}(M_1 \times M_2, k)^*$.

Universal property of the tensor product and categories

DEFINITION: Initial object of a category $\mathcal{C}$ is an object $X \in \mathcal{O}b(\mathcal{C})$ such that for any $Y \in \mathcal{O}b(\mathcal{C})$ there exists a unique morphism $X \longrightarrow Y$.

EXAMPLE: Zero space is an initial object in the category of vector spaces. The ring $\mathbb{Z}$ is an initial object in the category of rings with unit.

EXERCISE: Prove that **initial object is unique.**

DEFINITION: Let M_1, M_2 are R-modules, and $\mathcal{C}$ the following category. Objects of $\mathcal{C}$ are pairs (R-module M, bilinear map $M_1 \times M_2 \longrightarrow M$). Morphisms of $\mathcal{C}$ are homomorphisms $M \xrightarrow{\varphi} M'$ making the following diagram commutative:

$$\begin{array}{ccc} M_1 \times M_2 & \longrightarrow & M \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \varphi} \\ M_1 \times M_2 & \longrightarrow & M' \end{array}$$

CLAIM: (Universal property of the tensor product)
Tensor product $M_1 \times M_2$ is the initial object in $\mathcal{C}$.

COROLLARY: Tensor product is uniquely determined by the universal property.

Indeed, the initial object is unique.

Exactness of the tensor product (reminder)

THEOREM: Let $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. **Then the sequence**

$$M_1 \otimes_R M \longrightarrow M_2 \otimes_R M \longrightarrow M_3 \otimes_R M \longrightarrow 0 \qquad (*)$$

is exact.

Proof: Follows from the universal property and left exactness of the interior $\mathcal{H}om$ functors. ∎

COROLLARY: Let $I \subset R$ be an ideal in a ring. **Then** $M \otimes_R (R/I) = M/IM$**.**

Proof: Apply the functor $\otimes_R M$ to the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$. We obtain $IM \longrightarrow M \longrightarrow (R/I) \otimes_R M \longrightarrow 0$. ∎

Tensor product of rings (reminder)

DEFINITION: Let A, B be rings, $C \longrightarrow A$, $C \longrightarrow B$ homomorphisms. Consider A and B as C-modules, and let $A \otimes_{\mathbb{C}} B$ be their tensor product. Define the ring multiplication on $A \otimes_C B$ as $a \otimes b \cdot a' \otimes b' = aa' \otimes bb'$. This defines **tensor product of rings**.

EXAMPLE: $\mathbb{C}[t_1, ..., t_k] \otimes_{\mathbb{C}} \mathbb{C}[z_1, ..., z_n] = \mathbb{C}[t_1, ..., t_k, z_1, ..., z_n]$. Indeed, if we denote by $\mathbb{C}_d[t_1, ..., t_k]$ the space of polynomials of degree d, then $\mathbb{C}_d[t_1, ..., t_k] \otimes_{\mathbb{C}} \mathbb{C}_{d'}[z_1, ..., z_n]$ is polynomials of degree d in $\{t_i\}$ and d' in $\{z_i\}$.

EXAMPLE: For any homomorphism $\varphi : \mathbb{C} \longrightarrow A$, **the ring $A \otimes_C (C/I)$ is a quotient of A by the ideal $A \cdot \varphi(I)$**. This follows from $M \otimes_R (R/I) = M/IM$.

PROPOSITION: (associativity of $\otimes$)
Let $C \longrightarrow A, C \longrightarrow B, C' \longrightarrow B, C' \longrightarrow D$ be ring homomorphisms. Then $(A \otimes_C B) \otimes_{C'} D = A \otimes_C (B \otimes_{C'} D)$.

Proof: Universal property of $\otimes$ implies that $\operatorname{Hom}((A \otimes_C B) \otimes_{C'} D, M) = \operatorname{Hom}(A \otimes_C (B \otimes_{C'} D), M)$ is the space of polylinear maps $A \otimes B \otimes D \longrightarrow M$ satisfying $\varphi(ca, b, d) = \varphi(a, cb, d)$ and $\varphi(a, c'b, d) = \varphi(a, b, c'd)$. However, an object X of category is defined by the functor $\operatorname{Hom}(X, \cdot)$ uniquely **(prove it)**.
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Tensor products of rings: some examples (reminder)

DEFINITION: Recall that **the spectrum** of a finitely generated ring R is the corresponding algebraic variety, denoted by Spec(R)

PROPOSITION: Let $f: \ X \longrightarrow Y$ be a morphism of affine varieties, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ the corresponding ring homomorphism, $y \in Y$ a point, and $\mathfrak{m}_y$ its maximal ideal. **Denote by R_1 the quotient of $R := \mathcal{O}_X \otimes_{\mathcal{O}Y} (\mathcal{O}_Y/\mathfrak{m}_y)$ by its nilradical. Then Spec$(R_1) = f^{-1}(y)$.**

THEOREM: Let A, B be finitely generated reduced rings over $\mathbb{C}$. **Then $A \otimes_{\mathbb{C}} B$ is reduced, and Spec$(A \otimes_{\mathbb{C}} B) = $ Spec$(A) \times$ Spec(B).**

Preimage and diagonal

Claim 2: Let $f: X \longrightarrow Y$ be a morphism of algebraic varieties, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ the corresponding ring homomorphism, $Z \subset Y$ a subvariety, and I_Z its ideal. Denote by R_1 the quotient of a ring $R := \mathcal{O}_X \otimes_{\mathcal{O}Y} (\mathcal{O}_Y / I_Z) = \mathcal{O}_X / f^*(I_Z)$ by its nilradical. **Then** $\operatorname{Spec}(R_1) = f^{-1}(Z)$.

Proof: Clearly, the set of common zeros of the ideal $J := f^*(I_Z)$ contains $f^{-1}(Z)$. On the other hand, for any point $x \in X$ such that $f(x) \notin Z$ there exist a function $g \in J$ such that $g(x) \neq 0$. Therefore, $f^{-1}(Z) = V_J$, and strong Nullstellensatz implies that $\mathcal{O}_{f^{-1}(Z)} = R_1$. ■

Claim 3: Let M be an algebraic variety, and $\Delta \subset M \times M$ the diagonal, and $I \subset \mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M$ the ideal generated by $r \otimes 1 - 1 \otimes r$ for all $r \in \mathcal{O}_M$. **Then** $\mathcal{O}_\Delta$ **is** $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M / I$.

Proof. Step 1: By definition of the tensor product, $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M / I = \mathcal{O}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M = \mathcal{O}_M$, hence it is reduced. If we prove that $\Delta = V_I$, the statement of the claim would follow from strong Nullstellensatz.

Step 2: Clearly, $\Delta \subset V_I$. To prove the converse, let $(m, m') \in M \times M$ be a point not on diagonal, and $f \in \mathcal{O}_M$ a function which satisfies $f(m) = 0, f(m') \neq 0$. Then $f \otimes 1 - 1 \otimes f$ is non-zero on (m, m'). ■

Fibered product

DEFINITION: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be maps of sets. **Fibered product** $X \times_M Y$ is the set of all pairs $(x,y) \in X \times Y$ such that $\pi_X(x) = \pi_Y(y)$.

CLAIM: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be morphism of algebraic varieties, $R := \mathcal{O}_X \otimes_{\mathcal{O}_M} \mathcal{O}_Y$, and R_1 the quotient of R by its nilradical. **Then** $\operatorname{Spec}(R_1) = X \times_M Y$.

Proof: Let I be the ideal of diagonal in $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M$. Since I is generated by $r \otimes 1 - 1 \otimes r$ (Claim 3), $R = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_Y / (\pi_X \times \pi_Y)^*(I)$. Applying Claim 2, we obtain that $\operatorname{Spec}(R_1) = (\pi_X \times \pi_Y)^{-1}(\Delta)$. ■

Initial and terminal objects

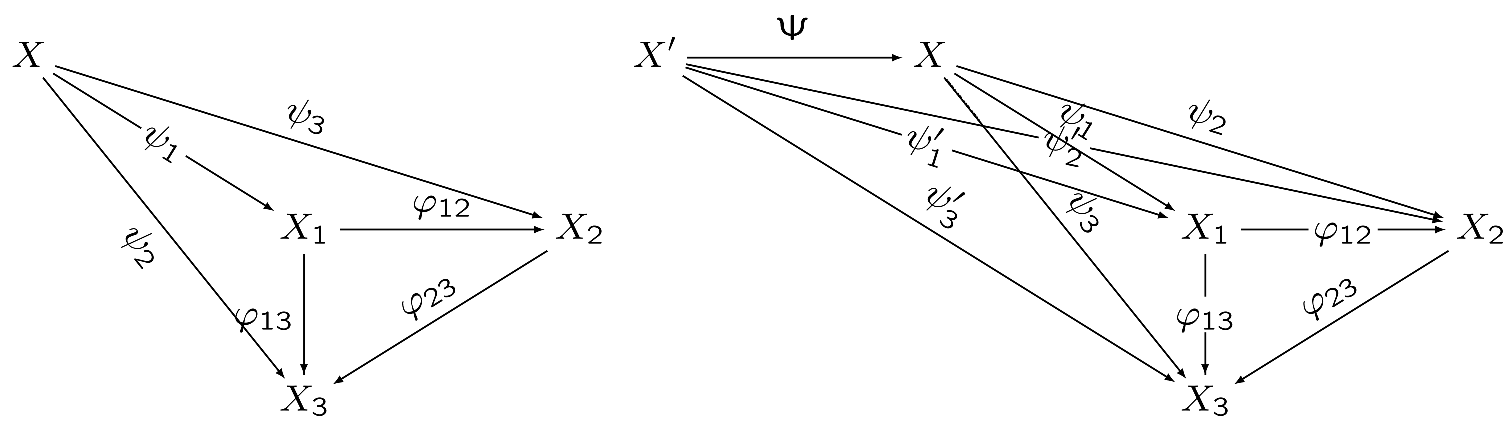
DEFINITION: Commutative diagram of vector spaces is given by the following data. There is a directed graph (graph with arrows). For each vertex of this graph we have a vector space, and each arrow corresponds to a homomorphism of the associated vector spaces. **These homomorphism are compatible, in the following way.** Whenever there exist two ways of going from one vertex to another, the compositions of the corresponding arrows are equal.

DEFINITION: An initial object of a category is an object $I \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(I, X)$ is always a set of one element. **A terminal object** is $T \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(X, T)$ is always a set of one element.

EXERCISE: Prove that **the initial and the terminal object is unique,** up to isomorphism.

Limits and colimits of diagrams

DEFINITION: Let $S = \{X_i, \varphi_{ij}\}$ be a commutative diagram in $\mathcal{C}$, and $\vec{\mathcal{C}}_S$ be a category of pairs (object X in $\mathcal{C}$, morphisms $\psi_i: X \longrightarrow X_i$, defined for all X_i) making the diagram formed by $(X, X_i, \psi_i, \varphi_{ij})$ commutative.



Morphisms $\mathcal{M}or(\{X, \psi_i\}, \{X', \psi'_i\})$, are morphisms $\Psi \in \mathcal{M}or(X, X')$, making the diagram formed by $(X, X', \psi_i, \psi'_i, \varphi_{ij})$ commutative. The terminal object in this category is called **limit**, or **inverse limit** of the diagram S.

DEFINITION: Colimit, or **direct limit** is obtained from the previous definition by inverting all arrows and replacing "terminal" by "initial".

Products and coproducts

EXAMPLE: Let S be a diagram with two vertices X_1 and X_2 and no arrows. The inverse limit of S is called **product** of X_1 and X_2, and inverse limit **the coproduct**.

EXAMPLE: Products in the category of sets, vector spaces and topological spaces are the usual products of sets, vector spaces and topological spaces **(check this)**.

EXAMPLE: Coproduct in the category of groups is called **free product**, or **amalgamated product**. Coproduct of the group $\mathbb{Z}$ with itself is called **free group**. Coproduct in the category of vector spaces is also the usual product of vector spaces.

Products and coproducts (2)

EXERCISE: Prove that **the product of algebraic varieties is their product in this category.**

EXERCISE: Prove that **coproduct of rings over $\mathbb{C}$ in the category of rings is their tensor product.**

EXERCISE: Prove that **coproduct of reduced rings over $\mathbb{C}$ in the category of reduced rings is the quotient of their tensor product over a nilradical.**

Since the category of algebraic varieties is equivalent to the category of finitely generated reduced rings, this gives another proof of the theorem.

THEOREM: Let A, B be finitely generated reduced rings over $\mathbb{C}$. **Then** $\operatorname{Spec}(A \otimes_{\mathbb{C}} B/I) = \operatorname{Spec}(A) \times \operatorname{Spec}(B)$, where I is nilradical.

Fibered product

DEFINITION: Consider the following diagram:

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd}$$

Its limit is called **fibered product** of A and B over C. Colimit of the diagram

$$\begin{tikzcd} A & & B \\ & C \arrow[ul] \arrow[ur] & \end{tikzcd}$$

is called **coproduct** of A and B over C.

EXERCISE: Prove that the **fibered product of algebraic varieties is the same as their product in the category of algebraic varieties.**

EXERCISE: Prove that the **coproduct of rings A and B over C is $A \otimes_C B$.** Prove that the **coproduct of reduced rings A and B over C in the category of reduced rings $A \otimes_C B/I$, where I is nilradical.**

Using strong Nullstellensatz again, we obtain

CLAIM: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be morphism of algebraic varieties, $R := \mathcal{O}_X \otimes_{\mathcal{O}_M} \mathcal{O}_Y$, and R_1 the quotient of R by its nilradical. **Then** $\operatorname{Spec}(R_1) = X \times_M Y$.

Zariski topology

DEFINITION: Zariski topology on an algebraic variety is a topology, where closed sets are algebraic subsets. **Zariski closure** of $Z \subset M$ is an intersection of all Zariski closed subsets containing Z.

DEFINITION: Cofinite topology is the topology on a set S such that the only closed subsets are S and finite sets.

EXERCISE: Prove that **Zariski topology on $\mathbb{C}$ coincides with the cofinite topology.**

REMARK: The same is true for $\mathbb{Z}$.

CAUTION: Zariski topology is non-Hausdorff.

Zariski topology (2)

REMARK: We defined the Zariski topology on the set of points of A, that is, on the set of maximal ideals of $\mathcal{O}_A$ (this is how Zariski defined it). **Following Grothendieck, one defines the Zariski topology on the set $\operatorname{Spec}_{pr}(\mathcal{O}_A)$ of all prime ideals in $\mathcal{O}_A$: closed subsets Z_I in this topology correspond to ideals $I \subset \mathcal{O}_A$, prime ideal $\mathfrak{p}$ lies in the closed subset Z_I if it contains I.**

Oscar Zariski
(1899 – 1986)

Dominant morphisms

DEFINITION: Dominant morphism is a morphism $f: X \longrightarrow Y$, such that Y is a Zariski closure of $f(X)$.

PROPOSITION: Let $f: X \longrightarrow Y$ be a morphism of affine varieties. **The morphism f is dominant if and only ig the homomorphism $\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X$ is injective**.

Proof. Step 1: If f^* is not injective, $f(X)$ lies in the set of common zeros of the ideal $\ker f^*$. Indeed, points of X are the same as maximal ideals and the same as homomorphisms $\mathcal{O}_X \longrightarrow \mathbb{C}$, and the points of $f(X)$ are homomorphisms $\mathcal{O}_Y \longrightarrow \mathbb{C}$ obtained as a composition $\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X \longrightarrow \mathbb{C}$.

Step 2: If $f(X)$ is contained in the set of common zeros of the ideal $J \subset \mathcal{O}_Y$, all functions $\alpha \in J$ vanish on $f(X)$. **This implies that $f^*(\alpha) = 0$.**

Field of fractions

DEFINITION: Let $S \subset R$ be a subset of R, closed under multiplication and not containing 0. **Localization** of R in S is a ring, formally generated by symbols a/F, where $a \in R$, $F \in S$, and relations $a/F \cdot b/G = ab/FG$, $a/F + b/G = \frac{aG+bF}{FG}$ and $aF^k/F^{k+n} = a/F^n$.

DEFINITION: Let R be a ring without zero divisors, and S the set of all non-zero elements in R. **Field of fractions** of R is a localization of R in S.

CLAIM: Let $f: X \longrightarrow Y$ be a dominant morphism, where X is irreducible. **Then Y is also irreducible**. Moreover, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ **can be extended to a homomorphism of the fields of fractions. $k(Y) \longrightarrow k(X)$.**

Proof. Step 1: Since $\mathcal{O}_Y$ is embedded in $\mathcal{O}_X$, and the later has no zero divisors, $\mathcal{O}_Y$ **has no zero divisors, hence Y is irreducible**.

Step 2: Embedding of rings without zero divisors can be extended to the fields of fractions: $f^*(a/F) = f^*(a)/f^*(F)$. ■