MATH-F-303 (Algebra and Geometry II), first term exam

Rules: Every student receives from me a list of 10 exercises (chosen randomly), and has to solve as many of them as you can by January 17. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solutions, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes). Please contact me by email `verbit2000[]gmail.com` when you are ready. The final score for this exam is $4k + 1/5s$, where k is the total number of points you got for the exam and s the sum of points for class exercises during the term.

1 Tensor product

Exercise 1.1. Consider the natural map (defined in lectures)

$$(V \otimes W)^* \longrightarrow \operatorname{Bil}_k(V \times W, k),$$

where $\operatorname{Bil}_k(V \times W, k)$ denotes bilinear maps. Prove that it is an isomorphism for infinite-dimensional spaces.

Exercise 1.2. Define a natural homomorphism

$$\operatorname{Hom}(V, \operatorname{Hom}(W, U)) \longrightarrow \operatorname{Hom}(V \otimes W, U),$$

in such a way that it is an isomorphism for all finite-dimensional spaces. Prove that it is an isomorphism for infinite-dimensional spaces, too.

Exercise 1.3. Define a natural homomorphism

$$\operatorname{Hom}(V_1, W_1) \otimes \operatorname{Hom}(V_2, W_2) \longrightarrow \operatorname{Hom}(V_1 \otimes V_2, W_1 \otimes W_2)$$

in such a way that it is an isomorphism for all finite-dimensional spaces. Prove that it is an isomorphism when V_1, W_1 are finite-dimensional, but V_2, W_2 are not necessarily finite-dimensional.

Exercise 1.4. Consider the natural map (defined in lectures)

$$V \otimes W^* \longrightarrow \operatorname{Hom}(W, V).$$

Prove that it is an isomorphism when either V or W is finite-dimensional. Prove that it is a not an isomorphism when both V and W are infinite-dimensional.

Exercise 1.5. Consider the natural map (defined in lectures)

$$V^* \otimes W^* \longrightarrow (V \otimes W)^*.$$

Prove that it is an isomorphism when either V or W is finite-dimensional. Prove that it is a not an isomorphism when both V and W are infinite-dimensional.

Definition 1.1. **Tensor product of algebras** $A \otimes B$ is the tensor product of A and B as vector spaces, with product in $A \otimes B$ defined by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

Exercise 1.6. Let V, W be vector spaces (not necessarily finite-dimensional). Prove that the algebra $\mathsf{Sym}^*(V) \otimes \mathsf{Sym}^*(W)$ is naturally isomorphic to $\mathsf{Sym}^*(V \oplus W)$).

Definition 1.2. Rank of a tensor $\psi \in V^{\otimes n}$ is a smallest number of tensor monomials $v_1 \otimes v_2 \otimes ... \otimes v_n$ such that ψ can be represented by a sum of these monomials.

Exercise 1.7. Prove that rank of a tensor $\psi \in V \otimes V^*$ is equal to the rank of the endomorphism which corresponds to this tensor under the map $V \otimes V^* \longrightarrow \operatorname{End} V$.

Exercise 1.8. Let $\dim V = d$. Prove that the maximal rank of a tensor $\psi \in V^{\otimes n}$ is d^n.

Exercise 1.9. Let $\psi \in V^{\otimes n}$ be a non-zero antisymmetric tensor. Find the minimal rank of ψ.

Exercise 1.10. Let V be a 2-dimensonal vector space over $\mathbb{R}$. Find a tensor of rank 3 in $V^{\otimes 3}$ or prove it does not exist.

2 Algebras and field extensions

Definition 2.1. Degree of an algebraic number $\alpha \in \mathbb{C}$ is degree of a smallest polynomial with coefficients in $\mathbb{Q}$ which has root α.

Exercise 2.1. Let α, β be algebraic numbers of degree p, q. Prove that $\alpha\beta$ and $\alpha + \beta$ has degree $\leqslant pq$.

Exercise 2.2. Find degree of $\sqrt{2} + \sqrt{3}$.

Exercise 2.3. Prove that the number $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

Exercise 2.4. Prove that the number $\sqrt{2} + \sqrt[3]{5}$ is irrational.

Exercise 2.5. Let $A \in \operatorname{End}(V)$ be an endomorphism, and $\mathbb{Q}[A]$ a subalgebra in $\operatorname{End}(V)$ generated by A. Prove that $\mathbb{Q}[A]$ is at most $\dim V$-dimensional.

Exercise 2.6. Consider the algebra $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ with product defined by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. Prove that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \operatorname{Mat}(2, \mathbb{C})$, where $\operatorname{Mat}(2, \mathbb{C})$ is algebra 2x2-matrices.

Definition 2.2. Let V be a vector space with bilinear symmetric form $g : V \otimes V \longrightarrow \mathbb{R}$. Recall that **Clifford algebra** $\operatorname{Cl}(V)$ is generated by V and defined by relations $v_1 \cdot v_2 + v_2 \cdot v_1 = g(v_1, v_2) \cdot 1$ for all $v_1, v_2 \in V$.

Exercise 2.7. Let $V = \mathbb{R}^2$ with scalar product of signature (1,1). Prove that $\operatorname{Cl}(V) \cong \operatorname{Mat}(2, \mathbb{R})$.

Exercise 2.8. Let $V = \mathbb{R}^2$ with positive definite scalar product. Prove that $\mathrm{Cl}(V) \cong \mathrm{Mat}(2, \mathbb{R})$.

Exercise 2.9. Let $V = \mathbb{R}^3$ with negative definite scalar product. Prove that $\mathrm{Cl}(V)$ contains zero divisors.

3 Grassmann algebra

Exercise 3.1. Let $\eta \in \mathrm{Bil}(V \times V, \mathbb{R})$ be a bilinear form for which $B(x,y) = 0 \Leftrightarrow= B(y,x) = 0$. Prove that η is symmetric or skew-symmetric.

Exercise 3.2. Let $\eta \in \Lambda^k V$ be a non-zero form, and $L_\eta : \quad \Lambda^1 \longrightarrow \Lambda^{1+k} V$ the multiplication map $x \longrightarrow x \wedge \eta$. Prove that its kernel $\ker L_\eta$ is at most k-dimensional.

Exercise 3.3. Let $V = \mathbb{R}^{2n}$, and $\alpha \in \Lambda^n V$, $\alpha \neq 0$. Prove that there exists $\beta \in \Lambda^n V$ such that $\alpha \wedge \beta \neq 0$.

Exercise 3.4. Let $V = \mathbb{R}^{2n}$, and $\alpha \in \Lambda^2 V^*$. Assume that $\alpha^n \in \Lambda^{2n} V^*$ is non-zero. Prove that α is a symplectic 2-form.

Exercise 3.5. Let $\omega_1, \omega_2 \in \Lambda^2(\mathbb{R}^4)$ symplectic forms satisfying $\omega_1 \wedge \omega_2 = 0$. Prove that $(\omega_1 + \omega_2) \wedge (\omega_2 - \omega_1) = 0$ or find a counterexample.

Exercise 3.6. Let V, W be vector spaces (not necessarily finite-dimensional). Prove that the vector space $\Lambda^*(V) \otimes \Lambda^*(W)$ is naturally isomorphic to $\Lambda^*(V \oplus W)$.

Exercise 3.7. Let $\omega \in \Lambda^2 V$ be a symplectic 2-form, $\dim V = 2n$. Prove that multiplication by ω defines an isomorphism $\Lambda^{n-1}(V) \longrightarrow \Lambda^{n+1}(V)$.

4 Orthogonal group and representation theory

Exercise 4.1. Let $A \in SO(4)$ be an orthogonal transform with one eigenvalue 1. Prove that there exists a quaternion $h \in \mathbb{H}$ such that A can be represented by a map $x \longrightarrow h^{-1} x h$ after identification $\mathbb{R}^4 = \mathbb{H}$.

Exercise 4.2. Consider the action of $SO(4)$ on a 2-dimensional space $\Lambda^2(\mathbb{R}^4)$ induced by action on $\mathbb{R}^4$ via $A(x \wedge y) = A(x) \wedge A(y)$. Decompose $\Lambda^2(\mathbb{R}^4)$ to a direct sum of two 3-dimensional subrepresentations.

Exercise 4.3. Construct an element $x \in SO(1,2)$ which is not diagonalizable (over $\mathbb{C}$).

Definition 4.1. Let V be a vector space with scalar product, and $A \in \mathrm{End}(V)$. **Polar decomposition** of A is $A = UP$, where U is orthogonal and P is positive self-adjoint.

Exercise 4.4. Let V be a vector space with scalar product. An operator $A \in \operatorname{End}(V)$ is called **normal** if $AA^* = A^*A$. Prove that A is normal if and only if in the polar decomposition $A = UP$, the operators U and P commute.

Exercise 4.5 (2 points). Let $V = \mathbb{R}^3$, and g, h two bilinear symmetric forms of signature (1,2). Prove that there exists a basis x_1, x_2, x_3 which is orthogonal with respect to g, h, or find a counterexample.

Exercise 4.6. Let G be a finite group and $k[G]$ the group algebra, which is a vector space of linear combinations $\sum_{g_i \in G} a_i g_i$ and the product induced by the product in G. Prove that any irreducible representation of G is a quotient representation of $k[G]$. Prove, moreover, that for $\operatorname{char} k = 0$, any irreducible representation of G is a subrepresentation of $k[G]$.

Exercise 4.7 (2 points). Let V be a representation of a finite group G over $\mathbb{R}$, and $\operatorname{End}_G(V)$ the algebra of G-invariant endomorphisms. Find G and V such that $\operatorname{End}_G(V)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.

5 Matrix algebra

Exercise 5.1. Let $A \in \operatorname{End}(\mathbb{R}^n)$ be a self-adjoint endomorphism. Prove that there exists a unique self-adjoint endomorphism B such that $B^3 = A$.

Exercise 5.2. Let $A \in \operatorname{End}(\mathbb{R}^n)$ be a strictly positive self-adjoint endomorphism. Prove thet there exists a unique self-adjoint endomorphism B such that $e^B = A$.

Exercise 5.3. Let $A \in \operatorname{End}(\mathbb{R}^n)$ be a diagonalizable endomorphism with pairwise nonequal eigenvalues. Prove that the set of solutions of equation $B^2 = A$ is finite, or find a counterexample.

Exercise 5.4 (2 points). Let $A \in \operatorname{End}(\mathbb{R}^n)$ be any endomorphism. Prove that the set of solutions of equation $B^2 = A$ is finite or uncountable.

Exercise 5.5. Suppose that $\operatorname{Tr}(A^n) = 0$ for all integers $n > 0$. Prive that A is nilpotent.

Exercise 5.6 (2 point). Inner automorphism of an algebra R is $X \longrightarrow AXA^{-1}$, where $A \in R$ is a fixed invertible element. Prove that all automorphisms of the matrix algebra are inner.

Exercise 5.7. Prove that the matrix algebra contains no non-trivial two-sided ideals.

Exercise 5.8. Let $A \in \operatorname{End} V$ be a diagonal operator on a vector space over k, and B an operator which commutes with any $C \in \operatorname{End} V$ satisfying $CA = AC$. Prove that $B = P(A)$ for some polynomial $P(t) \in k[t]$.