Algebra and Geometry

lecture 1: tensor product

Misha Verbitsky

Université Libre de Bruxelles

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Linear forms

DEFINITION: Let V be a vector space over k. **Linear form** on V is a homomorphism of vector spaces (that is, a linear map) $V \longrightarrow k$. The space of linear forms on V is called **the dual vector space** and denoted by V^*.

DEFINITION: **Kronecker symbol** δ_{ij} is equal to 1 if $i = j$ and to 0 otherwise.

CLAIM: **Dimensions of V and V^* are equal if V is finite-dimensional.**

Proof: Let $\{x_1, ..., x_n\}$ be a basis in V. Define **the dual basis** $\{\lambda_i\}$ in V^* by $\lambda_i(x_j) = \delta_{ij}$. **These two bases have the same number of elements.** ■

EXERCISE: Consider the natural map $V \stackrel{ev}{\to} V^{**}$, mapping a vector V to the linear form on V^* represented by v: $v \mapsto (\lambda \mapsto \lambda(v))$. **Prove that $V \stackrel{ev}{\to} V^{**}$ is an isomorphism when V is finite-dimensional.**

REMARK: It is **never** an isomorphism when V is infinite-dimensional **(prove it!)**

Bilinear maps

DEFINITION: Let U, V, W be vector spaces over k. A map $U \times V \stackrel{\mu}{\to} W$, $u, v \mapsto \mu(u,v)$ is called **bilinear** if for all $u \in U$ and $v \in V$ the maps $\mu(u,\cdot) : V \to W \quad \mu(\cdot, v) : U \to W$ are linear.

REMARK: Clearly, a linear combination of bilinear maps is bilinear. Then **the space** $\operatorname{Bil}(U \times V, W)$ **of bilinear maps** $U \times V \to W$ **is a vector space**.

DEFINITION: **Bilinear form** on V is a bilinear map $V \times V \stackrel{\mu}{\to} k$. **Bilinear symmetric form** is a form which satisfies $\mu(x,y) = \mu(y,x)$ for all $x, y \in V$. **Bilinear anti-symmetric formm** or **bilinear skew-symmetric form** is a form which satisfies $\mu(x,y) = -\mu(y,x)$. We denote the first by $\operatorname{Sym}^2 V^*$, and the second by $\Lambda^2 V^*$.

Symmetrization and antisymmetrization

From now on, we assume that $\operatorname{char} k \neq 2$.

DEFINITION: Consider the **symmetrization map** $\operatorname{Sym}(\mu)(x,y) = \frac{1}{2}(\mu(x,y) + \mu(y,x))$ from $\operatorname{Bil}(V \times V, k)$ to $\operatorname{Sym}^2(V^*)$ and **anti-symmetrization map** $\operatorname{Alt}(\mu)(x,y) = \frac{1}{2}(\mu(x,y) - \mu(y,x))$ from $\operatorname{Bil}(V \times V, k)$ to $\Lambda^2(V^*)$.

EXERCISE: Let A, B be vector subspaces in a vector space C. **Recall that** $C = A \oplus B$ means that $A \cap B = 0$ and C is generated by A, B. Prove that the map

$$\operatorname{Sym} \oplus \operatorname{Alt} : \operatorname{Bil}(V \times V, k) \to \operatorname{Sym}^2 V^* \oplus \Lambda^2 V^*$$

is an isomorphism.

EXERCISE: Let V be n-dimensional. **Find the dimension of** $\operatorname{Sym}^2 V^*$ **and** $\Lambda^2 V^*$**.**

Quadratic forms

Here we assume that $\operatorname{char} k \neq 2$.

DEFINITION: Let μ be a symmetric form on V. The map $q(v) := \mu(v,v)$ is called **quadratic form** on V.

CLAIM: Let q be a quadratic form associated with a bilinear symmetric form μ. **Suppose that $q = 0$. Then $\mu = 0$.**

Proof: Since $\mu(u+v, u+v) = \mu(v,v) + \mu(u,u) + 2\mu(u,v)$, one can express $\mu(u,v)$ in terms of q:

$$\mu(u,v) = \frac{q(u+v) - q(u) - q(v)}{2}.$$

■

EXERCISE: Find a non-zero bilinear symmetric form μ such that $q = 0$ for $\operatorname{char} k = 2$.

Non-degenerate forms

DEFINITION: Let V be a vector space, and μ a symmetric or antisymmertric bilinear form on it. Define **radical** $\ker\mu$ as the set of all vectors $v \in V$ such that $\mu(v, w) = 0$ for all $w \in V$.

REMARK: Since $\mu(v+r, v'+r') = \mu(v, v')$ for any $r, r' \in \ker\mu$, **this form is correctly defined on the quotient space $V/\ker\mu$.**

DEFINITION: Bilinear symmetric (or anti-symmetric) form is called **non-degenerate** if its radical is equal zero. A non-degenerate anti-symmetric form is called **symplectic**.

EXERCISE: Let μ be a non-degenerate symmetric (or anti-symmetric) bilinear form form on a finite-dimensional vector space V. **Prove that the map $v \mapsto \mu(v, \cdot)$ gives an isomorphism from V to V^*.**

Orthogonal complement

DEFINITION: Let μ be as above, and $V_1 \subset V$ a subspace. Assume that the restriction of μ to V_1 is non-degenerate. The **orthogonal complement** $V_1^\perp$ is the set of all $v \in V$ such that $\mu(v, v_1) = 0$ for all $v_1 \in V_1$ (that is, "v is orthogonal to V_1").

DEFINITION: Let A, B be vector subspaces in a vector space C. **We say that $C = A \oplus B$** if $A \cap B = 0$ and C is generated by A, B.

EXERCISE: Let μ be a non-degenerate form on V, and suppose that its resriction to V_1 is non-degenerate and $\dim V_1 < \infty$. **Prove that $V = V_1 \oplus V_1^\perp$.**

CLAIM: Let μ be a non-degenerate form on V, and suppose that its resriction to V_1 is non-degenerate and $\dim V_1 < \infty$. **Then $V = V_1 \oplus V_1^\perp$.**

Proof. Step 1: Clearly $V_1 \cap V_1^\perp = \ker \mu\big|_{V_1}$. **Since V_1 is non-degenerate, these spaces don't intersect.**

Step 2: Let $v \in V$, and consider $\mu(v, \cdot)$ as a functional on V_1. Since μ defines an isomorphism from V_1 to V_1^*, there exists $h \in V_1$ such that $\mu(v, \cdot) = \mu(h, \cdot)$ on V_1. Then $v - h \in V_1^\perp$, and v is decomposed as $v = h + (v - h)$. ∎

Orthogonal basis

DEFINITION: Let μ be a bilinear symmetric form on V. A basis $\{x_i\}$ is called **orthogonal** if $\mu(x_i, x_j) = 0$ for all $i \neq j$.

THEOREM: Let μ be a bilinear symmetric form on a finite-dimensional space V. **Then V admits an orthogonal basis.**

Proof. Step 1: Chose an arbitrary decompositon $V = \ker\mu \oplus V'$. Then μ is non-degenerate on V'. Choose an orthogonal basis $\{x_i\}$ in V' and any basis $\{y_i\}$ in $\ker\mu$. Then $\{x_i, y_i\}$ is an orthogonal basis in V. **It remains to show that V admits an orthogonal basis when μ is non-degenerate.**

Step 2: Since $\mu \neq 0$, there exists a vector v with $\mu(v, v) \neq 0$ (see the theorem about non-zero quadratic forms). This gives a decomposition $V = \langle v\rangle \oplus \langle v\rangle^\perp$. **Now we may use induction on $\dim V$ and obtain an orthogonal basis in V.** ■

REMARK: When $k = \mathbb{R}$, and μ non-degenerate, we may assume that $\mu(x_i, x_i) = \pm 1$ in the orthogonal basis. Such a basis is called **orthonormal**. When $k = \mathbb{C}$, we may assume $\mu(x_i, x_i) = 1$.

Symplectic basis

DEFINITION: Let V be a vector space equipped with a symplectic form. **Symplectic basis** is a basis $\{p_1, q_i\}$ such that $\mu(q_i, q_j) = \mu(p_i, p_j) = 0$ and $\mu(p_i, q_j) = \delta_{ij}$.

THEOREM: Let (V, μ) be a vector space equipped with a symplectic form. **Then V admits a symplectic basis.** In particular, **V is even-dimensional.**

Proof: Choose a non-zero vector $p_1 \in V$. Since $p_1 \notin \ker \mu$, there exists $q \in V$ such that $\mu(p_1, q) \neq 0$. Define $q_1 := q\mu(p_1, q)^{-1}$. Clearly, μ is non-degenerate on the space $\langle p_1, q_1 \rangle$ generated by p_1, q_1, and $\mu(p_1, q_1) = 1$. Using induction on $\dim V$, we find a symplectic basis $\{p_2, p_3, ..., p_n, q_2, q_3, ..., q_n\}$ in $\langle p_1, q_1 \rangle^\perp$. Then $\{p_1, p_2, p_3, ..., p_n, q_1, q_2, q_3, ..., q_n\}$ is a symplectic basis in V. ■

Tensor product

DEFINITION: Let S be a set. Define **vector space, freely generated by** S as the space of functions $\psi: S \longrightarrow k$ which are equal zero outside of a finite subset $\operatorname{Sup}_\psi \subset S$.

DEFINITION: Let V, V' be vector spaces over k, and W a vector space freely generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subspace generated by combinations $av\otimes v' - v\otimes av'$, $a(v\otimes v') - (av)\otimes v'$, $(v_1 + v_2)\otimes v' - v_1\otimes v' - v_2\otimes v'$ and $v \otimes (v_1' + v_2') - v \otimes v_1' - v \otimes v_2'$, where $a \in k$. Define **the tensor product** $V \otimes_k V'$ as a quotient vector space W/W_1.

PROPOSITION: For any vector spaces V, V', R, there is a natural identification $\operatorname{Hom}(V \otimes_k V', R) = \operatorname{Bil}(V \times V', R)$.

Proof: Clearly, any bilinear map $\rho \in \operatorname{Bil}(V \times V', R)$ defines a linear map $\tilde\rho: W \longrightarrow R$, and $\tilde\rho$ vanishes on W_1. This gives a map $\operatorname{Bil}(V\times V', R) \longrightarrow \operatorname{Hom}(V\otimes_k V', R)$. Inverse map takes $\tau \in \operatorname{Hom}(V \otimes_k V', R)$ and interprets it as a bilinear map in $\operatorname{Bil}(V \times V', R)$. ■

COROLLARY: For finite-dimensional V, V', one has $V \otimes_k V' = \operatorname{Bil}(V \times V', k)^*$.

Dimension of of tensor product

CLAIM: Dimension of $\operatorname{Bil}(V \times V', k)$ is equal to $\dim V \dim V'$.

Proof. Step 1: Let $\{\lambda_i\}$ be a basis in V^* and $\{\lambda'_i\}$ a basis in V'. Denote by $\{v_i\}$ $\{v'_i\}$ the dual basis in V, V'. Then $\lambda_i \lambda'_j$ can be interpreted as vectors in $\operatorname{Bil}(V \times V', k)$. These vectors are clearly linearly independent: indeed

$$\sum_{i,j} a_{ij} \lambda_i \lambda'_j (v_p, v'_q) = a_{pq}.$$

This gives $\dim \operatorname{Bil}(V \times V', k) \geqslant \dim V \dim V'$.

Step 2: On the other hand, $\dim V \otimes V' \leqslant \dim V \dim V'$, because it is generated by $v_p \otimes v_q$, hence $\dim \operatorname{Bil}(V \times V', k) \leqslant \dim V \dim V'$. ■

COROLLARY: Let $\{x_i\}$ and $\{y_i\}$ be bases in V, W. **Then $\{x_i \otimes y_j\}$ is a basis in $V \otimes_k W$.** ■

The space $\operatorname{End}(V)$

Consider the space $\operatorname{End}(V)$ of endomorphisms of a vector space V (that is, of linear maps from V to itself). Given $x \in V, \lambda \in V^*$, consider the map $x \otimes \lambda \in \operatorname{End}(V)$ mapping $y \in V$ to $x\lambda(y)$. This defines a bilinear map $\operatorname{Bil}(V \times V^*, \operatorname{End}(V))$. As usual, we associate with this map a homomorphism $\Psi : V \otimes_k V^* \longrightarrow \operatorname{End}(V)$.

THEOREM: The map $\Psi : V \otimes_k V^* \longrightarrow \operatorname{End}(V)$ constructed above **is an isomorphism for any finite-dimensional space V.**

Proof: The dimensions of $\operatorname{End}(V)$ and $V \otimes V^*$ are equal to n^2, hence it suffices to show that Ψ is surjective. However, elements $x \otimes \lambda \in \operatorname{End}(V)$ generate the space $\operatorname{End}(V)$ **(prove it).** ■