Algebra and Geometry

lecture 3: algebras defined by generators and relations

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Algebra over a field (reminder)

Fix a ground field k. Recall that a map $(V_1 \times V_2) \xrightarrow{\mu} V_3$ of vector spaces is called **bilinear** of for any $v_1 \in V_1$, $v_2 \in V_2$, the maps $\mu(v_1, \cdot) : V_2 \longrightarrow V_3$, $\mu(\cdot, v_2) : V_1 \longrightarrow V_3$ (one element is fixed) is k-linear.

To express this, we use the tensor product sign, and write $\mu : V_1 \otimes V_2 \longrightarrow V_3$.

DEFINITION: Let A be a vector space over k, and $\mu : A \otimes A \longrightarrow A$ a bilinear map (called **"multiplication"**). The pair (A, μ) is called **algebra over a field k** if μ is **associative**: $\mu(a_1, \mu(a_2, a_3)) = \mu(\mu(a_1, a_2), a_3))$. The product in algebra is written as $a \cdot b$ or ab. If, in addition, there is an element $1 \in A$ such that $\mu(1, a) = \mu(a, 1) = a$ for all $a \in A$, this element is called **unity**, a and A **an algebra with unity**.

DEFINITION: A **homomorphism** of algebras $r : A \longrightarrow A'$ is a linear map which is compatible with a product. **Isomorphism** of algebras is an invertible homomorphism. **Subalgebra** of an algebra A is a vector subspace which is closed under multiplication.

Algebraic and transcendental numbers

DEFINITION: Let $k \subset K$ be a field contained in a bigger field. In this case we say that k is **a subfield** of K, and K is **extension** of k. An element $x \in K$ is called **algebraic over** k if x is a root of a non-zero polynomial with coefficients in k.

DEFINITION: Algebraic number is an element of $\mathbb{C}$ which is algebraic over $\mathbb{Q}$. **Transcendental number** is an element which is not algebraic.

REMARK: The set of algebraic numbers is countable. Indeed, there is a countable number of non-zero polynomials in $\mathbb{Q}[t]$, each of them has finitely many roots, hence their roots also can be counted. Since $\mathbb{C}$ is non-countable, **there are a continually many transcendental numbers**.

Algebraic elements in a field extension

CLAIM: Let $x \in K$ be an element which is algebraic over $k \subset K$. **Then the algebra $k[x] := \langle 1, x, x^2, x^3, ..., x^n, ...\rangle$ generated by x is finite-dimensional over k.**

Proof: Indeed, if x is a root of $P(t) = t^n + \sum_{i=0}^{n-1} a_i t^i$, then x^n is expressed as a linear combination of smaller degrees: $x^n = -\sum_{i=0}^{n-1} a_i x^i$. Then the same is true for x^{n+1} and so on:

$$x^{n+1} = -\sum_{i=0}^{n-1} a_i x^{i+1} = -\sum_{i=1}^{n-1} a_i x^i + a_{n-1} \sum_{i=0}^{n-1} a_i x^i$$

and so on: **the algebra $k[x]$ is n-dimensional over k.** ■

The converse is also true, by the same argument:

CLAIM: Let $k \subset K$ and $x \in K$ be an element such that $k[x]$ is finite-dimensional over k. **Then x is algebraic over k.**

EXERCISE: Prove it!

COROLLARY: Suppose that x is algebraic over k. **Then all elements of $k[x]$ are algebraic over k.**

Finite extensions

PROPOSITION: Let $x \in K$ be an algebraic element over $k \subset K$. **Then $k[x]$ is a field.**

Proof: Multiplication by x defines an invertible map $L_x : k[x] \longrightarrow k[x]$, mapping z to xz. Since $k[x]$ is finite-dimensional, L_x is invertible. Then there exists $v \in k[x]$ such that $L_x(v) = 1$, giving $xv = 1$. ∎

DEFINITION: **A finite extension** is a field extension $K \supset k$ which is finite-dimensional as a vector space over k.

REMARK: As we just proved, **all elements of a finite extension are algebraic over k.**

Field of algebraic numbers

LEMMA: Let $K_1 \subset K_2 \subset K_3$ be fields such that K_2 is finite-dimensional over K_1, and K_3 is finite-dimensional over K_2. **Then K_3 is finite-dimensional over K_1.**

EXERCISE: **Prove it!**

PROPOSITION: **A sum and a product of two algebraic numbers is algebraic.**

Proof: Let α, β be algebraic over k elements of its extension $K \supset k$. Then $K_1 := k[\alpha]$ is a finite extension of k, and $K_2 := K_1[\beta]$ is finite over K_1. Applying the precious lemma, we obtain that $K_2 = k[\alpha, \beta]$ is finite-dimensional over k, and hence **all its elements are algebraic.** ■

REMARK: We obtained that the set of algebraic numbers in $\mathbb{C}$ is closed under all field operation (addition, substraction, multiplication, division), which defines **the field algebraic numbers**, denoted by $\overline{\mathbb{Q}}$.

EXERCISE: Using "fundamental theorem of algebra", prove that $\overline{\mathbb{Q}}$ is **algebraically closed**, that is, for each polynomial $P(t) \in \overline{Q}[t]$ of degree > 0, **the equation $P(t) = 0$ has solutions in $\overline{\mathbb{Q}}$.**

Algebra of polylinear forms

DEFINITION: Let V be a vector space over a field k. A **polylinear n-form**, or **n-linear form** φ on V is a map

$$\varphi: \underbrace{V \times V \times V \times \cdots \times V}_{n \text{ times}} \longrightarrow k,$$

linear in each argument. We write this as $\varphi: V \otimes V \otimes V \otimes \cdots \otimes V \longrightarrow k$. It is convenient to denote the space of n-linear forms as $(V^*)^{\otimes^n}$. In this notation, $(V^*)^{\otimes^0}$ is k (the ground field).

DEFINITION: Given polylinear j- and i-forms φ and ψ on V, the map $\varphi \otimes \psi : \underbrace{V \times V \times V \times \ldots}_{i+j} \longrightarrow k$ is given by

$$(\varphi \otimes \psi)(v_1, v_2, \ldots, v_{i+j}) = \varphi(v_1, \ldots, v_i)\varphi(v_{i+1}, \ldots, v_{i+j})$$

is clearly polylinear. This gives a multiplicative structure on the space $\bigoplus_{i=0}^{\infty}(V^*)^{\otimes^i}$, which is clearly associative. We call $\bigoplus_{i=0}^{\infty}(V^*)^{\otimes^i}$ the **algebra of polylinear forms.**

Tensor algebra

Let $V \times W$ map to $V \otimes W$ by putting v, w to $v \otimes w$. Clearly, this map is bilinear. Similarly, one has a bilinear map $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes m+n}$ putting $x_1 \otimes ... \otimes x_n, y_1 \otimes ... \otimes y_n$ to $x_1 \otimes ... \otimes x_n \otimes y_1 \otimes ... \otimes y_n$.

DEFINITION: Let V be a vector space over k. **Tensor algebra**, or **free algebra generated by V** is $T(V) := \bigoplus_i V^{\otimes^i}$ equipped with the multiplicative structure defined above,

EXERCISE: Prove that $T(V^*)$ **is the algebra of polylinear forms on V defined above.**

REMARK: If $x_1, ..., x_r$ is a basis in V, then **the basis in $V^{\otimes^n}$ is formed by all different monomials** of the form $x_{i_1} \otimes x_{i_2} \otimes ... \otimes x_{i_n}$.

Universal property of tensor algebra

CLAIM: Let $\varphi: V \longrightarrow A$ be a linear map from a vector space V to an algebra A (with unit). **Then φ can be uniquely extended to a homomorphism $\Phi: T(V) \longrightarrow A$** respecting a unit.

Proof. Step 1: Uniqueness is clear: indeed, **$T(V)$ is multiplicatively generated by V** (and unit).

Step 2: The vector space $T(V)$ is a quotient of the space $T_f(V)$ freely generated by the symbols $x_1 \otimes x_2 \otimes ... \otimes x_n$, $x_i \in V$ by the space $T_b(V)$ generated by "bilinear relations" of type

$$x_1 \otimes x_2 \otimes ... \otimes (x_i + x_i') \otimes ... \otimes x_n = x_1 \otimes x_2 \otimes ... \otimes x_i \otimes ... \otimes x_n + x_1 \otimes x_2 \otimes ... \otimes x_i' \otimes ... \otimes x_n.$$

and

$$x_1 \otimes x_2 \otimes ... \otimes ax_i \otimes ... \otimes x_n = a \cdot x_1 \otimes x_2 \otimes ... \otimes x_i \otimes ... \otimes x_n$$

The map φ is extended to $T_f(V)$ by putting

$$\Phi(x_1 \otimes x_2 \otimes ... \otimes x_n) = \varphi(x_1)\varphi(x_2)...\varphi(x_n).$$

This map vanishes on $T_w(V)$, because the product map $A \otimes A \otimes ... \otimes A \longrightarrow A$ satisfies the bilinear relations. ■

Two-sided ideals

DEFINITION: Let A be an algebra and $J \subset A$ its subspace. Then J is caled **left ideal** if for all $a \in A, j \in J$, one has $ja \in J$, and **right ideal** if one has $aj \in J$. J is called **two-sided ideal** if is is both right and left ideal.

REMARK: Let $J \subset A$ be a two-sided ideal, $x, y \in A/J$ some vectors, and $\tilde{x}, \tilde{y}$ Define the product $xy \in A/J$ by putting $x \cdot y$ to the class represented by $\tilde{x}, \tilde{y}$. Since $ja \in J$ and $aj \in J$, **this gives a bilinear map** $A/J \otimes A/J \longrightarrow A/J$, defining an associative multiplicative structure on A/J.

CLAIM: In these assumptions, A/J **is an algebra,** with the product defined as above.

EXERCISE: Prove it.

Algebra defined by generators and relations

DEFINITION: Let V be a vector space over k **("the space of generators")**, and $W \subset T(V)$ another vector space **("the space of relations")**. Consider a quotient A of $T(V)$ by the subspace $T(V)WT(V)$ generated by the vectors $v \otimes w \otimes v'$, where $w \in W$ and $v, v' \in T(V)$.

CLAIM: There is a natural product structure on the space $A := \frac{T(V)}{T(V)WT(V)}$.

Proof: $T(V)WT(V)$ is a 2-sided ideal. ■

DEFINITION: In this situation, we say that A **is an algebra defined by generators and relations**.

EXERCISE: Prove that any algebra can be defined by generators and relations.

Finitely presented algebras

DEFINITION: An algebra is called **finitely generated** if it can be defined by generators and relations, and the space of generators is finitely-dimensional. An algebra is called **finitely presented** if the space W of relations is finitely-dimensional.

EXERCISE: Find an algebra which is finitely-generated, but not finitely presented.

EXERCISE: Prove that **any finitely-dimensional algebra is finitely presented.**

EXERCISE: Represent the algebra of Laurent polynomials $k[t, t^{-1}]$ by generators and relations.

EXERCISE: Represent the polynomial algbera $k[x, y]$ by generators and relations.

Clifford algebras

Let V be a vector space with bilinear symmetric form $g: V \otimes V \longrightarrow \mathbb{R}$. Consider the algebra Cl(V) generated by V and defined by relations

$$v_1 \cdot v_2 + v_2 \cdot v_1 = g(v_1, v_2) \cdot 1,$$

for all $v_1, v_2 \in V$. This algebra is called **a Clifford alfebra** over k.

EXERCISE: Represent complex numbers as a Clifford algebra over $\mathbb{R}$.

EXERCISE: Represent quaternions as a Clifford algebra over $\mathbb{R}$.