Algebra and Geometry

lecture 4: polynomials and Grassmann algebra

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Algebra over a field (reminder)

Fix a ground field k. Recall that a map $(V_1 \times V_2) \xrightarrow{\mu} V_3$ of vector spaces is called **bilinear** of for any $v_1 \in V_1$, $v_2 \in V_2$, the maps $\mu(v_1, \cdot): V_2 \longrightarrow V_3$, $\mu(\cdot, v_2): V_1 \longrightarrow V_3$ (one element is fixed) is k-linear.

To express this, we use the tensor product sign, and write $\mu: V_1 \otimes V_2 \longrightarrow V_3$.

DEFINITION: Let A be a vector space over k, and $\mu: A \otimes A \longrightarrow A$ a bilinear map (called **"multiplication"**). The pair (A, μ) is called **algebra over a field k** if μ is **associative**: $\mu(a_1, \mu(a_2, a_3)) = \mu(\mu(a_1, a_2), a_3))$. The product in algebra is written as $a \cdot b$ or ab. If, in addition, there is an element $1 \in A$ such that $\mu(1, a) = \mu(a, 1) = a$ for all $a \in A$, this element is called **unity**, a and A **an algebra with unity**.

DEFINITION: A **homomorphism** of algebras $r: A \longrightarrow A'$ is a linear map which is compatible with a product. **Isomorphism** of algebras is an invertible homomorphism. **Subalgebra** of an algebra A is a vector subspace which is closed under multiplication.

Tensor algebra (reminder)

Let $V \times W$ map to $V \otimes W$ by putting v, w to $v \otimes w$. Clearly, this map is bilinear. Similarly, one has a bilinear map $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes m+n}$ putting $x_1 \otimes ... \otimes x_n, y_1 \otimes ... \otimes y_n$ to $x_1 \otimes ... \otimes x_n \otimes y_1 \otimes ... \otimes y_n$.

DEFINITION: Let V be a vector space over k. **Tensor algebra**, or **free algebra generated by V** is $T(V) := \bigoplus_{i=0}^{\infty} V^{\otimes^i}$ equipped with the multiplicative structure defined above, Here $V^{\otimes^0} = k$ (the ground field) and contains unit.

CLAIM: Let $\varphi : V \longrightarrow A$ be a linear map from a vector space V to an algebra A (with unit). **Then φ can be uniquely extended to a homomorphism** $\Phi : T(V) \longrightarrow A$ respecting a unit.

Two-sided ideals (reminder)

DEFINITION: Let A be an algebra and $J \subset A$ its subspace. Then J is caled **left ideal** if for all $a \in A, j \in J$, one has $ja \in J$, and **right ideal** if one has $aj \in J$. J is called **two-sided ideal** if is is both right and left ideal.

REMARK: Let $J \subset A$ be a two-sided ideal, $x, y \in A/J$ some vectors, and $\tilde{x}, \tilde{y}$ Define the product $xy \in A/J$ by putting $x \cdot y$ to the class represented by $\tilde{x}, \tilde{y}$. Since $ja \in J$ and $aj \in J$, **this gives a bilinear map** $A/J \otimes A/J \longrightarrow A/J$, defining an associative multiplicative structure on A/J.

CLAIM: In these assumptions, A/J **is an algebra,** with the product defined as above. Conversely, the for any surjective algebra homomorphism $A \longrightarrow A_1$, its kernel is a 2-sided ideal.

EXERCISE: **Prove it.**

Algebra defined by generators and relations (reminder)

DEFINITION: Let V be a vector space over k **("the space of generators")**, and $W \subset T(V)$ another vector space **("the space of relations")**. Consider a quotient A of $T(V)$ by the subspace $T(V)WT(V)$ generated by the vectors $v \otimes w \otimes v'$, where $w \in W$ and $v, v' \in T(V)$.

CLAIM: **There is a natural product structure on the space $A := \frac{T(V)}{T(V)WT(V)}$.**

Proof: $T(V)WT(V)$ is a 2-sided ideal. ■

DEFINITION: In this situation, we say that A **is an algebra defined by generators and relations**.

EXERCISE: **Prove that any algebra can be defined by generators and relations.**

DEFINITION: An algebra is called **finitely generated** if it can be defined by generators and relations, and the space of generators is finitely-dimensional. An algebra is called **finitely presented** if the space W of relations is finitely-dimensional.

Clifford algebras

Let V be a vector space with bilinear symmetric form $g: V \otimes V \longrightarrow \mathbb{R}$. Consider the algebra $\mathrm{Cl}(V)$ generated by V and defined by relations

$$v_1 \cdot v_2 + v_2 \cdot v_1 = g(v_1, v_2) \cdot 1,$$

for all $v_1, v_2 \in V$. This algebra is called **a Clifford alfebra** over k.

EXERCISE: Represent complex numbers as a Clifford algebra over $\mathbb{R}$.

EXERCISE: Represent quaternions as a Clifford algebra over $\mathbb{R}$.

Graded algebras

DEFINITION: An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead of $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always in A^0, so that it is non-empty.

EXAMPLE: The tensor algebra $T(V)$ and the polynomial algebra $\operatorname{Sym}^*(V)$ are obviously graded.

DEFINITION: A subspace $W \subset A^*$ of a graded algebra is called **graded** if W is a direct sum of components $W^i \subset A^i$.

EXERCISE: Let $W \subset T(V)$ be a graded subspace. Prove that then **the algebra generated by V with relation space W is also graded.**

Symmetric algebra

DEFINITION: Consider a subspace $W \subset V \otimes V$ generated by vectors $x \otimes y - y \otimes x$. Then the algebra $\operatorname{Sym}^*(V) := \frac{T(V)}{T(V)WT(V)}$ is commutative **(check this)**. Since W is a graded subspace, $\operatorname{Sym}^*(V)$ is a graded algebra.

CLAIM: For any commutative algebra A over k and any linear map $\varphi : V \longrightarrow A$, **$\varphi$ can be uniquely extended to an algebra homomorphism $\Phi : \operatorname{Sym}^*(V) \longrightarrow A$.**

Proof. Step 1: Clearly, φ can be extended to a homomorphism $\Phi_t : T^*V \longrightarrow A$ from the tensor algebra T^*V.

Step 2: Since A is commutative, $\Phi_t(xy) = \Phi_t(yx)$. Therefore, Φ_t vanishes on the ideal $T(V)WT(V)$. ■

COROLLARY: Let V be a vector space over k with basis $x_1, ..., x_n$. Then **$\operatorname{Sym}^* V$ is isomorphic to the polynomial algebra $k[x_1, ..., x_n]$.**

Proof: Consider the linear map $V = \langle x_1, ..., x_n \rangle \longrightarrow k[x_1, ..., x_n]$ mapping a vector to the corresponding homogeneous degree 1 polynomial. This map can be extended to a homomorphism $\operatorname{Sym}^* V \longrightarrow k[x_1, ..., x_n]$ as shown above. Inverse map takes $x_i \in k[x_1, ..., x_n]$ to the corresponding vector $x_i \in V$. ■

The Grassmann algebra

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a subspace generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

REMARK: Grassmann algebra is a a Clifford algebra with the symmetric form $g = 0$ (for $g \neq 0$ Clifford algebra is no longer graded).

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

Signature of a permutation

DEFINITION: The group Σ_n acts on the polynomial ring $P[x_1, ..., x_n]$ by permutation of variables. Consider the polynomial $P(x_1, ..., x_n) := \prod_{i<j}(x_i - x_j)$. Clearly, for any permutation $\sigma \in \Sigma_n$, we have $\sigma(P) = \pm P$. **This defines a homomorphism** $\operatorname{sign} : \Sigma_n \longrightarrow \{\pm 1\}$**, with** $\sigma(P) = \operatorname{sign}(\sigma)P$**.**

REMARK: This homomorphism maps a product of odd number of transpositions to -1 and a product of even number of transpositions to 1.

DEFINITION: The number $\operatorname{sign}(\sigma)$ is called **signature** of a permutation σ. Permutation σ is called **odd** if $\operatorname{sign}(I) = -1$ and **even** if $\operatorname{sign}(\sigma) = 1$. For odd permutation σ we write $\tilde{\sigma} := 1$, for even permutation, $\tilde{\sigma} = 0$.

Antisymmetric tensors

DEFINITION: Let $V^{\otimes^n}$ be n-th product of V with itself, equipped with the natural symmetric group Σ_n-action exchanging the tensor components. A tensor $\psi \in V^{\otimes^n}$ is called **antisymmetric** if for any permutation $\sigma \in \Sigma_n$ we have $\sigma(\psi) = (-1)^{\tilde\sigma} psi$, and **symmetric** if $\sigma(\psi) = \psi$. We denote the space of all antisymmetric tensors by $\tilde\Lambda^n V$ and the space of symmetric tensors by $\widetilde{\operatorname{Sym}}^n V$.

Theorem 1: Let V be a vector space over a field of char $= 0$, $\Lambda^n V$ the n-th component of its Grassmann algebra, and $\operatorname{Sym}^n V$ the n-th component of its symmetric algebra. Then **$\Lambda^n V$ is naturally identified with $\tilde\Lambda^n V$, and $\operatorname{Sym}^n V$ with $\widetilde{\operatorname{Sym}}^n V$.**

Proof: See later.

Representations of finite groups

DEFINITION: Let V be a vector space. We denote by $GL(V)$ the group of all invertible linear maps from V to itself. It is called **the matrix group**, or **the group of linear automorphisms** of V.

DEFINITION: Let G be a group, V a vector space, and $\rho : \ G \longrightarrow GL(V)$ a group homomorphism. Then ρ is called **representation of G**, and V **the space of representation ρ**. We often denote the action $v \mapsto \rho(g)(v)$ by $v \mapsto g(v)$.

EXAMPLE: Symmetric group Σ_n acting on the direct sum V^n of n copies of W by exchanging its summands.

EXAMPLE: Symmetric group Σ_n acting on the tensor product $W^{\otimes^n}$ n copies of W by exchanging the tensor components.

Invariants and coinvariants

DEFINITION: Let $\rho: G \longrightarrow GL(V)$ be a group representation. **The space of invariants** V^G is the space of all vectors $v \in V$ which are **invariant** under G-action, that is, satisfy $g(v) = v$ for all $g \in G$. **The space of coinvariants** V_G is the quotient of V by its subspace generated by $v - g(v)$ for all $v \in V$, $g \in G$.

EXAMPLE: By definition, $\operatorname{Sym}^n V$ **is the space of coinvariants of Σ_n-action on $V^{\otimes n}$**, and $\widetilde{\operatorname{Sym}}^n V$ **is the space of invariants** of this action: $\operatorname{Sym}^n V = (V^{\otimes^n})_{\Sigma_n}$, $\widetilde{\operatorname{Sym}}^n V = (V^{\otimes^n})^{\Sigma_n}$.

EXAMPLE: Let $A_n \subset \Sigma_n$ be the group of all even permutations. Then the group $\Sigma_n/A_n = \{\pm 1\}$ acts on the space of A_n-invariants and A_n-coinvariants. **Anti-invariants** of this groups are vectors where its generator acts as -1. Clearly, $\widetilde{\Lambda}^n V$ **is the space of Σ_n/A_n-anti-invariant vectors in $(V^{\otimes^n})^{A_n}$, and $\Lambda^n V$ is the space of Σ_n/A_n-anti-invariant vectors in $(V^{\otimes^n})_{A_n}$.**

REMARK: To prove Theorem 1, **it would suffice to identify invariants and coinvariants** of Σ_n (for symmetric tensors) and A_n for antisymmetric.

Symmetrization

Proposition 1: Let G be a finite group acting on a vector space V over a field of char = 0, and $\Pi: V \longrightarrow V_G$ the natural projection to coinvariants. **Then the Π induces an isomorphism of invariants and coinvariants $\Pi: V^G \longrightarrow V_G$.**

Proof. Step 1: Consider **the symmetrization map** $S_G(v) := \frac{1}{|G|}\sum_{g\in G} g(v)$. Since G exchanges the summands in the sum $\sum_{g\in G} g(v)$, **all vectors in the image of S_G are G-invariant.**

Step 2: Clearly, $\ker S_G$ contains all vectors of form $v - g(v)$, hence S_G defines a map from coinvariants to invariants: $S_G: V_G \longrightarrow V^G$, and $S_G(\Pi(v)) = v$ for all G-invariant vectors $v \in V^G$.

Step 3: For any coinvariant $w \in V_G$, represented by $\tilde w \in V$,

$$\Pi(S_G(w)) = \Pi\left(\frac{1}{|G|}\sum_{g\in G} g(\tilde w)\right) = w$$

hence **$S_G: V_G \longrightarrow V^G$ is inverse to $\Pi: V^G \longrightarrow V_G$.** ■

EXERCISE: Find a counterexample to this proposition for an infinite group G.

Antisymmetrization

Similarly, the natural map from $\tilde{\Lambda}^n V$ to $\Lambda^n V$ is given by **antisymmetrization**

$$\operatorname{Alt}(x_1 \otimes ... \otimes x_n) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{\tilde{\sigma}} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}).$$

Its properties:

1. Clearly, $\operatorname{im}\operatorname{Alt}$ lies in the space of antisymmetric tensors, and $\operatorname{Alt}(\eta - (-1)^{\tilde{\sigma}}\eta) = 0$, hence Alt defines a map from Grassmann algebra to the space of antisymmetric tensors.

2. The natural projection from antisymmetric tensors to the Grassmann algebra is inverse to Alt **(check this; the argument is literally the same as proves isomorphism of invariants and coinvariants)**. Therefore, **the natural projection from antisymmetric tensors to $\Lambda^i V$ is an isomorphism.**

We proved Theorem 1 for antisymmetric tensors; for symmetric tensors it follows directly from Proposition 1.

Grassmann algebra and determinant

Grassmann algebra is "algebra generated by anticommuting variables", similarly to polynomial algebra which is generated by commuting variables.

EXERCISE: Prove that $\dim \Lambda^i V = \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.

REMARK: Let W be a one-dimensional vector space over k. **Then** $\operatorname{End} W$ **is naturally isomorphic to** k**.**

REMARK: Let $A \in \operatorname{End}(V)$ be a linear endomorphism of a vector space V. Then **the action of** A **on** $V \cong \Lambda^1 V$ **is uniquely extended to a multiplicative homomorphism of the algebra** $\Lambda^* V$**.**

DEFINITION: Let V be a d-dimensional vector space and $A \in \operatorname{End}(V)$. Consider the induced endomorphism of the space of determinant vectors $\Lambda^d(V)$ denoted as $\det A \in \operatorname{End}(\Lambda^d(V))$. Since $\Lambda^d(V)$ is 1-dimensional, the space $\operatorname{End}(\Lambda^d(V))$ is naturally identified with k. **This allows to consider** $\det A$ **as a number, that is, an element of** k**.** This number is called **determinant** of A.

REMARK: From the definition it is clear that det **defines a homomorphism from the group** $GL(V)$ **if invertible matrices to the multiplicative group** k^* **of the ground field.**