Algebra and Geometry

lecture 5: group representations

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Group representations

DEFINITION: Let V be a vector space. We denote by $GL(V)$ the group of all invertible linear maps from V to itself. It is called **the matrix group**, or **the group of linear automorphisms** of V.

DEFINITION: Let G be a group, V a vector space, and $\rho: G \longrightarrow GL(V)$ a group homomorphism. Then ρ is called **representation of G**, and V **the space of representation ρ**. We often denote the action $v \mapsto \rho(g)(v)$ by $v \mapsto g(v)$. Two representations V, W are **isomorphic** if there exists a linear isomorphism $\varphi: V \longrightarrow W$ compatible with G-action.

EXAMPLE: Symmetric group Σ_n acting on the direct sum V^n of n copies of W by exchanging its summands.

EXAMPLE: Symmetric group Σ_n acting on the tensor product $W^{\otimes^n}$ of n copies of W by exchanging the tensor components.

REMARK: **All vector spaces are assumed finite-dimensional** unless otherwise specified.

Group algebra

EXAMPLE: Let $k[G]$ be a vector space with basis $g_1, g_2, ...$, where g_i are all $g \in G$. We define multiplication on $k[G]$ as follows:

$$\left(\sum a_i g_i\right)\left(\sum b_i g_i\right) = \sum_{i,j} a_i b_j g_i g_j.$$

Then $k[G]$ is called **the group algebra** of G. G acts on $k[G]$ in several ways:

1. **Left action:** $g\left(\sum a_i g_i\right) = \sum a_i g g_i$.

2. **Right action:** $g\left(\sum a_i g_i\right) = \sum a_i g_i g^{-1}$.

3. **Adjoint action:** $g\left(\sum a_i g_i\right) = \sum a_i g g_i g^{-1}$.

EXERCISE: Prove that $k[G]$ **with left action is not always isomorphic to** $k[G]$ **with adjoint action.**

EXERCISE: Prove that $k[G]$ **with left action is always isomorphic to** $k[G]$ **with right action** (as a G-representation).

Tensor product and dual representations

DEFINITION: Let V, W be representations of G. Then G acts on $V \otimes W$ as $g(x \otimes y) := g(x) \otimes g(y)$. The representation $V \otimes W$ is called **the tensor product of the representations V and W.**

REMARK: To see that this product is defined correctly, we consider the bilinear map $V \times W \longrightarrow V \otimes W$ defined by $g(x, y) := g(x) \otimes g(y)$, and extend it to the tensor product using the **universal property of tensor product.**

DEFINITION: Let V be a representation of G, and $\lambda \in V^*$. We define $g(\lambda)$ using $\langle g(\lambda), v\rangle := \langle \lambda, g^{-1}(v)\rangle$ for each $g \in G, v \in V$. It is called **the dual representation**.

DEFINITION: A **tensor space** of V is $V^{\otimes i} \otimes (V^*)^{\otimes j}$. From this construction we obtain an action of G on all tensor spaces.

Invariant scalar product

DEFINITION: Let V be a representation of a finite group G over a field k with $\operatorname{char} k = 0$. Consider **the symmetrization map** $S_G(v) := \frac{1}{|G|}\sum_{g\in G} g(v)$. Since G exchanges the summands in the sum $\sum_{g\in G} g(v)$, **all vectors in the image of S_G are G-invariant.**

CLAIM: Let k be a subfield of $\mathbb{R}$, V a vector space over k with action of a group G, h a positive definite scalar product with values in k, and $S_G(h)$ its symmetrization. **Then $S_G(h)$ is positive definite and G-invariant.**

EXERCISE: Prove it.

CLAIM: Let k be a subfield of $\mathbb{C}$, V a vector space over k with action of a group G, h a positive definite Hermitian form with values in k, and $S_G(h)$ its symmetrization. **Then $S_G(h)$ is positive definite and G-invariant.**

EXERCISE: Prove it.

Irreducible representations

DEFINITION: Let V be a representation of G. **Subrepresentation** is a subspace $W \subset V$ which is G-invariant. A representation is called **irreducible** if it has no subrepresentations, and **semisimple** if it is a direct sum of irreducible representations.

THEOREM: Let V be a finite-dimensional representation of a finite group over a field k, $\operatorname{char} k = 0$. **Then V is semisimple.**

Proof: Let h be a G-invariant positive definite scalar product (for $k \subset \mathbb{R}$) or a Hermitian form for $k \subset \mathbb{C}$, and $W \subset V$ a subrepresentation. Then $V = W \oplus W^\perp$. Since h is G-invariant, for any $g \in G$, $w \in W$, $w_1 \in W^\perp$, one has $h(w, g(w_1)) = h(g(w), w_1) = 0$, hence $W^\perp$ is G-invariant. We have decomposed V into a direct sum of two subrepresentations. Since the dimension of V is finite, this process cannot go forever, and in the end we shall decompose V into a direct sum of irreducibless. ■

DEFINITION: A decomposition of a representation into a sum of irreducible ones is called **the irreducible decomposition**.

Invariants and coinvariants

DEFINITION: Let $\rho: G \longrightarrow GL(V)$ be a group representation. **The space of invariants** V^G is the space of all vectors $v \in V$ which are **invariant** under G-action, that is, satisfy $g(v) = v$ for all $g \in G$. **The space of coinvariants** V_G is the quotient of V by its subspace generated by $v - g(v)$ for all $v \in V$, $g \in G$.

CLAIM: Let V be an irreducible representation of G. **Then $V^G = V_G = 0$ if V is non-trivial, and $V^G = V_G = V$ if it is trivial.**

Proof: To see that $V_G = 0$ for any non-trivial irreducible representation V, we notice that **the vectors $v - g(v)$ generate a non-trivial subprepresentation $V_1 \subset V$,** hence $V = V_1$. ■

COROLLARY: Let V be a representation of a finite group over a field of char $= 0$, and $\Pi: V \longrightarrow V_G$ the natural projection map. **Then $\Pi\big|_{V^G}$ defines an isomorphism of invariants and coinvariants.**

Proof: Let $V = \oplus V_i$ be an irreducible decomposition of V. Then $V_i^G = (V_i)_G = V_i$ when V_i is trivial, and $V_i^G = (V_i)_G = 0$ when V_i is non-trivial. ■

Graded algebras (reminder)

DEFINITION: An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead of $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always in A^0, so that it is non-empty.

EXAMPLE: The tensor algebra $T(V)$ and the polynomial algebra $\operatorname{Sym}^*(V)$ are obviously graded.

DEFINITION: A subspace $W \subset A^*$ of a graded algebra is called **graded** if W is a direct sum of components $W^i \subset A^i$.

EXERCISE: Let $W \subset T(V)$ be a graded subspace. Prove that then **the algebra generated by V with relation space W is also graded.**

Symmetric algebra (reminder)

DEFINITION: Consider a subspace $W \subset V \otimes V$ generated by vectors $x \otimes y - y \otimes x$. Then the algebra $\operatorname{Sym}^*(V) := \frac{T(V)}{T(V)WT(V)}$ is commutative **(check this)**. Since W is a graded subspace, $\operatorname{Sym}^*(V)$ is a graded algebra.

CLAIM: For any commutative algebra A over k and any linear map $\varphi : V \longrightarrow A$, **$\varphi$ can be uniquely extended to an algebra homomorphism $\Phi : \operatorname{Sym}^*(V) \longrightarrow A$.**

Proof. Step 1: Clearly, φ can be extended to a homomorphism $\Phi_t : T^*V \longrightarrow A$ from the tensor algebra T^*V.

Step 2: Since A is commutative, $\Phi_t(xy) = \Phi_t(yx)$. Therefore, Φ_t vanishes on the ideal $T(V)WT(V)$. ■

COROLLARY: Let V be a vector space over k with basis $x_1, ..., x_n$. Then **$\operatorname{Sym}^* V$ is isomorphic to the polynomial algebra $k[x_1, ..., x_n]$.**

Proof: Consider the linear map $V = \langle x_1, ..., x_n \rangle \longrightarrow k[x_1, ..., x_n]$ mapping a vector to the corresponding homogeneous degree 1 polynomial. This map can be extended to a homomorphism $\operatorname{Sym}^* V \longrightarrow k[x_1, ..., x_n]$ as shown above. Inverse map takes $x_i \in k[x_1, ..., x_n]$ to the corresponding vector $x_i \in V$. ■

The Grassmann algebra (reminder)

Grassmann algebra is "algebra generated by anticommuting variables", similarly to polynomial algebra which is generated by commuting variables.

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a subspace generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i \in \mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i \in \mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

Signature of a permutation

DEFINITION: The group Σ_n acts on the polynomial ring $P[x_1, ..., x_n]$ by permutation of variables. Consider the polynomial $P(x_1, ..., x_n) := \prod_{i<j}(x_i - x_j)$. Clearly, for any permutation $\sigma \in \Sigma_n$, we have $\sigma(P) = \pm P$. **This defines a homomorphism** $\operatorname{sign} : \Sigma_n \longrightarrow \{\pm 1\}$**, with** $\sigma(P) = \operatorname{sign}(\sigma)P$**.**

REMARK: This homomorphism maps a product of odd number of transpositions to -1 and a product of even number of transpositions to 1.

DEFINITION: The number $\operatorname{sign}(\sigma)$ is called **signature** of a permutation σ. Permutation σ is called **odd** if $\operatorname{sign}(I) = -1$ and **even** if $\operatorname{sign}(\sigma) = 1$. For odd permutation σ we write $\tilde{\sigma} := 1$, for even permutation, $\tilde{\sigma} = 0$.

EXERCISE: Let $\Pi : V^{\otimes k} \longrightarrow \Lambda^k V$ be a natural projection. **Prove that** $\ker \Pi$ **is generated by all vectors of form** $x_1 \otimes x_2 \otimes ... \otimes x_k - (-1)^{\tilde{\sigma}} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$**.**

Antisymmetric tensors

DEFINITION: Let $V^{\otimes^n}$ be n-th product of V with itself, equipped with the natural symmetric group Σ_n-action exchanging the tensor components. A tensor $\psi \in V^{\otimes^n}$ is called **antisymmetric** if for any permutation $\sigma \in \Sigma_n$ we have $\sigma(\psi) = (-1)^{\tilde\sigma}\psi$, and **symmetric** if $\sigma(\psi) = \psi$. We denote the space of all antisymmetric tensors by $\tilde\Lambda^n V$ and the space of symmetric tensors by $\widetilde{\operatorname{Sym}}^n V$.

Theorem 1: Let V be a vector space over a field of char = 0, $\Lambda^n V$ the n-th component of its Grassmann algebra, and $\operatorname{Sym}^n V$ the n-th component of its symmetric algebra. Then **$\Lambda^n V$ is naturally identified with $\tilde\Lambda^n V$, and $\operatorname{Sym}^n V$ with $\widetilde{\operatorname{Sym}}^n V$:** the projection from $V^{\otimes n}$ to $\Lambda^n V$ (or $\operatorname{Sym}^n V$) induces an isomorphism from $\tilde\Lambda^n V$ (or $\widetilde{\operatorname{Sym}}^n V$) to $\Lambda^n V$ (or $\operatorname{Sym}^n V$).

Proof. Step 1: $\operatorname{Sym}^n V$ is the space of coinvariants of Σ_n-action on $V^{\otimes n}$, and $\widetilde{\operatorname{Sym}}^n V$ is the space of invariants of this action. As we have seen, **these spaces are isomorphic.**

Antisymmetric tensors (2)

Theorem 1: Let V be a vector space over a field of char $=0$, $\Lambda^n V$ the n-th component of its Grassmann algebra, and $\operatorname{Sym}^n V$ the n-th component of its symmetric algebra. Then **$\Lambda^n V$ is naturally identified with $\tilde{\Lambda}^n V$, and $\operatorname{Sym}^n V$ with $\widetilde{\operatorname{Sym}}^n V$:** the projection from $V^{\otimes n}$ to $\Lambda^n V$ (or $\operatorname{Sym}^n V$) induces an isomorphism from $\tilde{\Lambda}^n V$ (or $\widetilde{\operatorname{Sym}}^n V$) to $\Lambda^n V$ (or $\operatorname{Sym}^n V$).

Step 2: Let A be a non-trivial 1-dimensional representation of symmetric group, with all $\sigma \in \Sigma_n$ acting as $(-1)^{\tilde{\sigma}}$. Consider an irreducible decomposition of $V^{\otimes n}$, $V^{\otimes n} = A^k \oplus \bigoplus W_i$, where W_i are irreducible representations not isomorphic to A. **Then $A^k \subset V^{\otimes n}$ is the space of antisymmetric tensors.**

Step 3: The kernel K of the projection $V^{\otimes n} \longrightarrow \Lambda^n V$ is generated by all vectors of form $v - (-1)^{\tilde{\sigma}} \sigma(v)$. In each of $W_i \ncong A$, this subspace is non-trivial, hence equal to the whole of W_i. Therefore, $K = \bigoplus W_i$, and the natural projection $\tilde{\Lambda}^n V \longrightarrow \Lambda^n V$ is an isomorphism. ■

Antisymmetrization (reminder)

The natural map from $\Lambda^n V$ to $\tilde{\Lambda}^n V$ is given by **antisymmetrization**

$$\operatorname{Alt}(x_1 \otimes ... \otimes x_n) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{\tilde{\sigma}} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}.$$

Its properties:

1. Clearly, $\operatorname{im} \operatorname{Alt}$ lies in the space of antisymmetric tensors, and $\operatorname{Alt}(\eta - (-1)^{\tilde{\sigma}}\eta) = 0$, hence Alt **defines a map from Grassmann algebra to the space of antisymmetric tensors.**

2. The natural **projection from antisymmetric tensors to the Grassmann algebra is inverse to** Alt**.**

Grassmann algebra: dimension of components

REMARK: For linearly independent vectors $x_1, ..., x_k$, the antisymmetrization $x_1 \wedge x_2 \wedge ... \wedge x_k := \frac{1}{k!} \sum (-1)^{\tilde\sigma} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ is non-trivial. Indeed, the monomials $x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ are linearly independent in $V^{\otimes k}$. **This implies** $\dim \Lambda^k V \geqslant \binom{\dim V}{k}$.

CLAIM: $\dim \Lambda^k V = \binom{\dim V}{k}$**, and** $\dim \Lambda^* V = 2^{\dim V}$**.**

Proof: Let $x_1, ..., x_n$ be a basis in V. Then **the space $\Lambda^k V$ is generated by antisymmetric tensors $x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_k}$, $i_1 < i_2 < ... < i_k$, which are all linearly independent.** ■

Grassmann algebra and determinant

REMARK: Let W be a one-dimensional vector space over k. **Then** $\operatorname{End} W$ **is naturally isomorphic to** k**.**

REMARK: Let $A \in \operatorname{End}(V)$ be a linear endomorphism of a vector space V. Then **the action of** A **on** $V \cong \Lambda^1 V$ **is uniquely extended to a multiplicative endomorphism of the algebra** $\Lambda^* V$**.**

DEFINITION: Let V be a d-dimensional vector space and $A \in \operatorname{End}(V)$. Consider the induced endomorphism of the space of determinant vectors $\Lambda^d(V)$ denoted as $\det A \in \operatorname{End}(\Lambda^d(V))$. Since $\Lambda^d(V)$ is 1-dimensional, the space $\operatorname{End}(\Lambda^d(V))$ is naturally identified with k. **This allows to consider** $\det A$ **as a number, that is, an element of** k**.** This number is called **determinant** of A.

REMARK: From this definition it is clear that $\det$ **defines a homomorphism from the group** $GL(V)$ **if invertible matrices to the multiplicative group** k^* **of the ground field.**