Algebra and Geometry

lecture 6: Grassman algebra revisited

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Group representations (reminder)

DEFINITION: Let V be a vector space. We denote by $GL(V)$ the group of all invertible linear maps from V to itself. It is called **the matrix group**, or **the group of linear automorphisms** of V.

DEFINITION: Let G be a group, V a vector space, and $\rho : G \longrightarrow GL(V)$ a group homomorphism. Then ρ is called **representation of G**, and V **the space of representation ρ**. We often denote the action $v \mapsto \rho(g)(v)$ by $v \mapsto g(v)$. Two representations V, W are **isomorphic** if there exists a linear isomorphism $\varphi : V \longrightarrow W$ compatible with G-action.

Irreducible representations (reminder)

DEFINITION: Let V be a representation of G. **Subrepresentation** is a subspace $W \subset V$ which is G-invariant. A representation is called **irreducible** if it has no subrepresentations, and **semisimple** if it is a direct sum of irreducible representations.

THEOREM: Let V be a finite-dimensional representation of a finite group over a field k, $\operatorname{char} k = 0$. **Then V is semisimple.**

DEFINITION: A decomposition of a representation into a sum of irreducible ones is called **the irreducible decomposition**.

Invariants and coinvariants (reminder)

DEFINITION: Let $\rho: G \longrightarrow GL(V)$ be a group representation. **The space of invariants** V^G is the space of all vectors $v \in V$ which are **invariant** under G-action, that is, satisfy $g(v) = v$ for all $g \in G$. **The space of coinvariants** V_G is the quotient of V by its subspace generated by $v - g(v)$ for all $v \in V$, $g \in G$.

CLAIM: Let V be an irreducible representation of G. **Then $V^G = V_G = 0$ if V is non-trivial, and $V^G = V_G = V$ if it is trivial.**

Proof: To see that $V_G = 0$ for any non-trivial irreducible representation V, we notice that **the vectors $v - g(v)$ generate a non-trivial subpreprentation $V_1 \subset V$,** hence $V = V_1$. ■

COROLLARY: Let V be a representation of a finite group over a field of char $= 0$, and $\Pi: V \longrightarrow V_G$ the natural projection map. **Then $\Pi\big|_{V^G}$ defines an isomorphism of invariants and coinvariants.**

Proof: Let $V = \oplus V_i$ be an irreducible decomposition of V. Then $V_i^G = (V_i)_G = V_i$ when V_i is trivial, and $V_i^G = (V_i)_G = 0$ when V_i is non-trivial. ■

The Grassmann algebra (reminder)

Grassmann algebra is "algebra generated by anticommuting variables", similarly to polynomial algebra which is generated by commuting variables.

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a subspace generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

Antisymmetric tensors (reminder)

DEFINITION: Let $V^{\otimes^n}$ be n-th product of V with itself, equipped with the natural symmetric group Σ_n-action exchanging the tensor components. A tensor $\psi \in V^{\otimes^n}$ is called **antisymmetric** if for any permutation $\sigma \in \Sigma_n$ we have $\sigma(\psi) = (-1)^{\tilde{\sigma}}\psi$, and **symmetric** if $\sigma(\psi) = \psi$. We denote the space of all antisymmetric tensors by $\tilde{\Lambda}^n V$ and the space of symmetric tensors by $\widetilde{\operatorname{Sym}}^n V$.

Theorem 1: Let V be a vector space over a field of char $= 0$, $\Lambda^n V$ the n-th component of its Grassmann algebra, and $\operatorname{Sym}^n V$ the n-th component of its symmetric algebra. Then **$\Lambda^n V$ is naturally identified with $\tilde{\Lambda}^n V$, and $\operatorname{Sym}^n V$ with $\widetilde{\operatorname{Sym}}^n V$:** the projection from $V^{\otimes n}$ to $\Lambda^n V$ (or $\operatorname{Sym}^n V$) induces an isomorphism from $\tilde{\Lambda}^n V$ (or $\widetilde{\operatorname{Sym}}^n V$) to $\Lambda^n V$ (or $\operatorname{Sym}^n V$).

Antisymmetrization (reminder)

The natural map from $\Lambda^n V$ to $\tilde{\Lambda}^n V$ is given by **antisymmetrization**

$$\operatorname{Alt}(x_1 \otimes ... \otimes x_n) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{\tilde{\sigma}} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}.$$

Its properties:

1. Clearly, $\operatorname{im} \operatorname{Alt}$ lies in the space of antisymmetric tensors, and $\operatorname{Alt}(\eta - (-1)^{\tilde{\sigma}} \eta) = 0$, hence Alt **defines a map from Grassmann algebra to the space of antisymmetric tensors.**

2. The natural **projection from antisymmetric tensors to the Grassmann algebra is inverse to** Alt**.**

Grassmann algebra: dimension of components (reminder)

REMARK: For linearly independent vectors $x_1, ..., x_k$, the antisymmetrization $x_1 \wedge x_2 \wedge ... \wedge x_k := \frac{1}{k!}\sum(-1)^{\tilde\sigma} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ is non-trivial. Indeed, the monomials $x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ are linearly independent in $V^{\otimes k}$. **This implies** $\dim \Lambda^k V \geqslant \binom{\dim V}{k}$.

CLAIM: $\dim \Lambda^k V = \binom{\dim V}{k}$, **and** $\dim \Lambda^* V = 2^{\dim V}$.

Proof: Let $x_1, ..., x_n$ be a basis in V. Then **the space $\Lambda^k V$ is generated by antisymmetric tensors $x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_k}$, $i_1 < i_2 < ... < i_k$, which are all linearly independent.** ■

Grassmann algebra and determinant (reminder)

REMARK: Let W be a one-dimensional vector space over k. **Then** $\operatorname{End} W$ **is naturally isomorphic to** k**.**

REMARK: Let $A \in \operatorname{End}(V)$ be a linear endomorphism of a vector space V. Then **the action of** A **on** $V \cong \Lambda^1 V$ **is uniquely extended to a multiplicative endomorphism of the algebra** $\Lambda^* V$**.**

DEFINITION: Let V be a d-dimensional vector space and $A \in \operatorname{End}(V)$. Consider the induced endomorphism of the space of determinant vectors $\Lambda^d(V)$ denoted as $\det A \in \operatorname{End}(\Lambda^d(V))$. Since $\Lambda^d(V)$ is 1-dimensional, the space $\operatorname{End}(\Lambda^d(V))$ is naturally identified with k. **This allows to consider** $\det A$ **as a number, that is, an element of** k**.** This number is called **determinant** of A.

REMARK: From this definition it is clear that $\det$ **defines a homomorphism from the group** $GL(V)$ **if invertible matrices to the multiplicative group** k^* **of the ground field.**

Perfect pairing

DEFINITION: Let V, W be vector spaces over k, and $h: V \times W \longrightarrow k$ a bilinear form. It is called **a perfect pairing**, or **non-degenerate**, if for any $x \in V$, there exists $y \in W$ such that $h(x, y) \neq 0$, and for any $y \in W$, there exists $x \in V$ such that $h(x, y) \neq 0$.

REMARK: Let $h: V \times W \longrightarrow k$, with V finitely dimensional. **Then W is finitely dimensional, and $V \cong W^*$.**

Proof: The pairing defines maps $h_1 : V \longrightarrow W^*$, $x \longrightarrow h(x, \cdot)$, and $h_2 : W \longrightarrow V^*$, $y \longrightarrow h(\cdot, y)$, **Both of these maps are injective** by definition of perfect pairing. Since $\dim V = \dim V^*$, this implies that $\dim V \leqslant \dim W \leqslant \dim V$, and both h_1 and x_2 are isomorphisms. ■

Poincaré duality for Grassmann algebra

DEFINITION: Let $\dim V = n$ be a vector space over k, and let det be a generator of a 1-dimensional space $\Lambda^n V$. Consider the following pairing $\bigwedge: \Lambda^p V \times \Lambda^{n-p} V \longrightarrow \Lambda^n V$, mapping x, y to $x \wedge y$, and let $p: \Lambda^p V \times \Lambda^{n-p} \longrightarrow k$ map x, y to $\frac{\bigwedge(x,y)}{\det}$. Then p is called **Poincaré pairing for Grassmann algebra**.

CLAIM: The Poincaré pairing $p: \Lambda^p V \times \Lambda^{n-p} V \longrightarrow k$ is non-degenerate.

Proof: Take the basis $x_1, ..., x_n$ in V. Then $x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_p}$, $i_1 < i_2 < ... < i_p$ is a basis in $\Lambda^p V$, $x_{j_1} \wedge x_{j_2} \wedge ... \wedge x_{j_{n-p}}$ $j_1 < j_2 < ... < j_{n-p}$ is a basis in $\Lambda^{n-p} V$, and

$$(x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_p}) \wedge (x_{j_1} \wedge x_{j_2} \wedge ... \wedge x_{j_{n-p}}) = \begin{cases} \pm 1, & \text{if } i_1, ..., i_p \text{ and } j_1, j_2, ..., j_{n-p} \text{ are complementary} \\ 0 & \text{otherwise.} \end{cases}$$

■

Duality and inverse matrix

CLAIM: Let $A \in \operatorname{End}(V)$ be an endomorphism of an n-dimensional space. **Then A is invertible if and only if $\det A \neq 0$.**

Proof. Step 1: If A is invertible, then $1 = \det(AA^{-1}) = \det(A)\det(A^{-1})$, hence $\det A$ is invertible.

Step 2: Conversely, suppose that $\det A$ is invertible. Extend A to an endomorphism of the Grassmann algebra: $A: \Lambda^* V \longrightarrow \Lambda^* V$. Let $\breve{A}$ denote the endomorphism of $\Lambda^{n-1} V$ induced by A. Then $\bigwedge(x, y) \det A = \bigwedge(A(x), \breve{A}(y))$, and the pairing $x, y \longrightarrow \bigwedge(A(x), \breve{A}(y))$ is non-degenerate. Then **for each non-zero $x \in V$, the form $\bigwedge(A(x), \cdot)$ is non-zero, hence $A(x) \neq 0$.** ■

Inverse matrix: explicit description

DEFINITION: Let $A \in \operatorname{End} V$ be an operator on a vector space. **Adjoint operator** is $A^t \in \operatorname{End}(V^*)$ such that for any $x \in V, y \in V^*$, one has $\langle A(x), \lambda\rangle = \langle x, A^t(\lambda)\rangle$

REMARK: If we chose a basis $x_1, ..., x_n$ in V and the dual basis $\lambda_1, ..., \lambda_n$ in V^*, then **the matrix of A^t is transposed matrix of A.**

CLAIM: Extend $A \in \operatorname{End}(A)$ to an endomorphism of the Grassmann algebra: $A: \Lambda^* V \longrightarrow \Lambda^* V$, where $\dim V = n$. Let $\breve{A}$ denote the endomorphism of $\Lambda^{n-1} V$ induced by A. Let us identify V^* and $\Lambda^{n-1} V$ by Poincare pairing, and denote by $\breve{A}^t$ the adjoint endomorphism of V. **Then $A = \det A^{-1} \breve{A}^t$.**

Proof: Let $x \in V$, $y \in \Lambda^{n-1} V$. Then $p(x,y) = p(A(x), \det A^{-1} \breve{A}(y))$, hence $p(x,y) = p(x, A^t \det A^{-1} \breve{A}(y))$. This implies that $A^t \det A^{-1} \breve{A}$ acts as identity on $V^* = \Lambda^{n-1} V$. Passing to adjoint operator, we obtain $A \det A^{-1} \breve{A}^t = \operatorname{Id}$. ■

Cayley-Hamilton theorem

DEFINITION: Let $A \in \operatorname{End} V$ be an endomorphism of a vector space. Consider $t\operatorname{Id} - A$ as endomorphism of V, and let $P(t) := \det(t\operatorname{Id} - A)$ be its determinant considered as a polynomial on t. Then $P(t)$ is called **characteristic polynomial** of A.

THEOREM: (Cayley-Hamilton)
Let $A \in \operatorname{End}(V)$, and $P(t)$ be its characteristic polynomial. Consider $P(A)$ as an endomorphism of A. **Then $P(A) = 0$.**

Proof: Let $Z(B) := \check{B}^t$. This is a polynomial function of matrix coefficients of B. Then $Z(t\operatorname{Id} - A) \in \operatorname{End}(V)[t]$ is an $\operatorname{End}(V)$-valued polynomial function of t. This gives $(t\operatorname{Id} - A)Z(t\operatorname{Id} - A) = \det(t\operatorname{Id} - A)\operatorname{Id} = P(t)\operatorname{Id}$. Plugging $t = A$ in this formula, we obtain $P(A) = 0$. ■