Algebra and Geometry

lecture 7: Lie groups

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Matrix exponent and Lie groups

DEFINITION: Exponent of an endomorphism A is $e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$.

EXERCISE: Prove that **if $A, B \in \operatorname{End}(V)$ commute, one has $e^{A+B} = e^A e^B$.**

EXERCISE: Find an example when $A, B \in \operatorname{End}(V)$ **do not commute, and $e^{A+B} \neq e^A e^B$.**

EXERCISE: Prove that **exponent is invertible in a sufficiently small neighbourhood of 0** (the proof will be given today).

DEFINITION: Let $W \subset \operatorname{End}(V)$ be a subspace such that $e^W := \bigcup_{w \in W} e^w$ is a subgroup of $GL(V)$. A group $G \subset GL(V)$ is called **Lie subgroup of $GL(V)$**, or **matrix Lie group**, if its connected component is e^W. In this case W is called its **Lie algebra**.

REMARK: Definition of "connected component" of a group will be given later today.

Lie groups: first examples

EXAMPLE: From invertibility of exponent it follows that $GL(V) = e^W$, for $W = \text{End}(V)$ **(prove it)**.

EXERCISE: Prove that $\det e^A = e^{\operatorname{Tr} A}$, where $\operatorname{Tr} A$ is a trace of A.

EXAMPLE: Let $SL(V)$ be the group of all matrices with determinant 1, and $\text{End}_0(V)$ the space of all matrices with trace 0. Then $e^{\text{End}_0(V)} = SL(V)$ **(prove it).** This implies that **$SL(V)$ is also a Lie group.**

Topological spaces: remedial definitions

DEFINITION: A set of all subsets of M is denoted 2^M. **Topology** on M is a collection of subsets $S \subset 2^M$ called **open subsets**, and satisfying the following conditions.

* **Empty set and M are open**
* **A union of any number of open sets is open**
* **An intersection of a finite number of open subsets is open.**

A complement of an open set is called **closed**.

EXAMPLE: Let M be a metric space. A subset $U \subset M$ is called **open** if it is obtained as a union of open balls. This topology is called **induced by the metric**.

DEFINITION: A map $\varphi : M \longrightarrow M'$ of topological spaces is called **continuous** if a preimage of each open set $U \subset M'$ is open in M.

DEFINITION: Let M, N be topological spaces. **Product topology** is a topology on $M \times N$, with open sets obtained as unions $\bigcup_\alpha U_\alpha \times V_\alpha$, where U_α is open in M and V_α is open in N.

DEFINITION: Let M be a topological space, and $Z \subset M$ its subset. **Open subsets** of Z are subsets obtained as $Z \cap U$, where U is open in M. This topology is called **induced topology**.

Connected spaces

DEFINITION: A topological space is called **connected** if it cannot be partitioned into a union of non-intersecting non-empty open subsets. It is called **disconnected** otherwise.

EXERCISE: Prove that **the product of two connected spaces is connected.**

EXERCISE: Prove that a union of two intersecting connected subsets $Z_1, Z_2 \subset M$ **is connected.**

Claim 1: Let $\varphi: M_1 \longrightarrow M_2$ be a surjective, continuous map of topological spaces. Suppose that M_1 is connected. **Then M_2 is also connected.**

Proof: If M_2 is partitioned as a union of non-intersecting open subsets, $M_2 = U \coprod V$, **then $M_1 = \varphi^{-1}(U) \coprod \varphi^{-1}(V)$, hence also disconnected.** ■

DEFINITION: Let $m \in M$ be a point of a topological space. **Connected component** of $m \in M$ is the union of all connected subsets $Z \subset M$ containing m.

Path connected spaces

EXERCISE: Prove that **a line $\mathbb{R}$ and any interval $[a,b] \subset \mathbb{R}$ is connected.**

DEFINITION: A path in a topological space M is a continuous map from an interval to M.

COROLLARY: Let $\gamma: [a,b] \longrightarrow M$ be a path in M. Then **the image of γ belongs to a connected component of M.**

DEFINITION: A topological space M is called **path connected** if any two points of M can be connected by a path.

REMARK: A path connected space is connected.

Topological groups

DEFINITION: **Topological group** is a group G equipped with topology in such a way that the group operations $G \times G \xrightarrow{\mu} G$ (multiplication) and $G \xrightarrow{x \mapsto x^{-1}} G$ are continuous.

CLAIM: Let G be a topological group, and G_0 the connected component of unit in G. **Then $G_0 \subset G$ is a subgroup.**

Proof: We need only to show that the maps $G \times G \xrightarrow{\mu} G$ and $G \xrightarrow{x \mapsto x^{-1}} G$ preserve G_0. However, an image of a connected space is connected, hence **$\mu(G_0 \times G_0)$ and inverse of G_0 belong to G_0,** ■

DEFINITION: This subgroup is called **connected component of G**.

Differentiable maps

DEFINITION: Let $U, V \subset \mathbb{R}^n$ be open subsets. **An affine map** is a sum of linear map α and a constant map. Its **linear part** is α.

DEFINITION: Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open subsets. A map $f: U \longrightarrow V$ is called **differentiable** if it can be approximated by an affine one at any point: that is, for any $x \in U$, there exists an affine map $\varphi_x: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that

$$\lim_{x_1 \to x} \frac{|f(x_1) - \varphi(x_1)|}{|x - x_1|} = 0$$

DEFINITION: Differential, or **derivative** of a differentiable map $f: U \longrightarrow V$ is the linear part of φ.

DEFINITION: Diffeomorphism is a differentiable map f which is invertible, and such that f^{-1} is also differentiable. A map $f: U \longrightarrow V$ is a **local diffeomorphism** if each point $x \in U$ has an open neighbourhood $U_1 \ni x$ such that $f: U_1 \longrightarrow f(U_1)$ is a diffeomorphism.

REMARK: Chain rule says that a composition of two differentiable functions is differentiable, and its differential is composition of their differentials.

Banach fixed point theorem

LEMMA: (Banach fixed point theorem/"contraction principle")
Let $U \subset \mathbb{R}^n$ be a closed subset, and $f: U \longrightarrow U$ a map which satisfies $|f(x)-f(y)| < k|x-y|$, where $k<1$ is a real number (such a map is called **"contraction"**). **Then f has a fixed point, which is unique.**

Proof. Step 1: Uniqueness is clear because for two fixed points x_1 and x_2 $|f(x_1)-f(x_2)| = |x_1-x_2| < k|x_1-x_2|$.

Step 2: Existence follows because the sequence $x_0 = x, x_1 = f(x), x_2 = f(f(x)), ...$ satisfies $|x_i - x_{i+1}| \leqslant k|x_{i-1}-x_i|$ which gives $|x_n - x_{n+1}| < k^n a$, where $a = |x-f(x)|$. Then $|x_n - x_{n+m}| < \sum_{i=0}^m k^{n+i} a \leqslant k^n \frac{1}{1-k} a$, hence $\{x_i\}$ is a Cauchy sequence, and converges to a limit y, which is unique.

Step 3: $f(y)$ is a limit of a sequence $f(x_0), f(x_1), ... f(x_i), ...$ which gives $y = f(y)$. ■

EXERCISE: Find a counterexample to this statement when U is open and not closed.

Inverse function theorem

THEOREM: Let $U, V \subset \mathbb{R}^n$ be open subsets, and $f: U \longrightarrow V$ a differentiable map. Suppose that the differential of f is everywhere invertible. **Then f is locally a diffeomorphism.**

Proof. Step 1: Let $x \in U$. Without restricting generality, we may assume that $x = 0$, $U = B_r(0)$ is an open ball of radius r, and **in U one has $\frac{|f(x_1)-\varphi(x_1)|}{|x-x_1|} < 1/2$.** Replacing f with $-f \circ (D_0 f)^{-1}$, where $D_0 f$ is differential of f in 0, **we may assume also that $D_0 f = -\operatorname{Id}$.**

Step 2: In these assumptions, $|f(x)+x| < 1/2|x|$, hence $\psi_s(x) := f(x)+x-s$ is a contraction. This map maps $\overline{B}_{r/2}(0)$ to itself when $s < r/4$. By Banach fixed point theorem, **$\psi_s(x) = x$ has a unique fixed point x_s, which is obtained as a solution of the equation $f(x)+x-s = x$, or, equivalently, $f(x) = s$.** Denote the map $s \longrightarrow x_s$ by g.

Step 3: By construction, $fg = \operatorname{Id}$. Applying the chain rule, we find that g is also differentiable. ■

Lie groups as submanifolds

DEFINITION: A subset $M \subset \mathbb{R}^n$ is **an m-dimensional smooth submanifold** if for each $x \in M$ there exists an open in $\mathbb{R}^n$ neighbourhood $U \ni x$ and a diffeomorphism from U to an open ball $B \subset \mathbb{R}^n$ which maps $U \cap M$ to an intersection $B \cap R^m$ of B and an m-dimensional linear subspace.

PROPOSITION: Let $G \subset \operatorname{End}(V)$ be a Lie subgroup in $GL(V)$. **Then G is a submanifold.**

Proof. Step 1: From inverse function theorem, it follows that $A \longrightarrow e^A$ is a diffeomorphism on a neighbourhood of 0 mapping the Lie algebra W of G to G. This gives a diffeomorphism from a sufficiently small open ball $U_0 \subset W$ to a neighbourhood U of $\operatorname{Id} = e^0$ in G.

Step 2: For any $g \in G$, consider the map $x \longrightarrow ge^x$. This map defines a diffeomorphism between a neighbourhood of 0 in $\operatorname{End}(V)$ and a neighbourhood gU of g, mapping W to $gU \subset G$. ■

Orthogonal group as a Lie group

DEFINITION: Let V be a vector space equipped with a non-degenerate bilinear symmetric form h. Then the group of all endomorphisms of V preserving h and orientation is called **orthogonal group**, denoted by $SO(V,h)$.

DEFINITION: Consider the space of all $A \in \operatorname{End}(V)$ which satisfy $h(Ax,y) = -h(x,Ay)$. This space is called **the space of antisymmetric matrices** and denoted $\mathfrak{so}(V,h)$.

REMARK: Clearly, $\mathfrak{so}(V,h) = \{A \in \operatorname{End}(V) \ \mid \ A^t = -A\}$.

THEOREM: $SO(V,h)$ **is a Lie group, and** $\mathfrak{so}(V,h)$ **its Lie algebra.**

Proof. Step 1:

$$0 = \frac{d}{dt} h(e^{tA}v, e^{tA}w) = h(Ae^{tA}v, e^{tA}w) + h(e^{tA}v, Ae^{tA}w).$$

If h is e^{tA}-invariant, this gives $0 = h(Av,w) + h(v,Aw)$, hence A is antisymmetric.

Orthogonal group as a Lie group (2)

THEOREM: **$SO(V,h)$ is a Lie group, and $\mathfrak{so}(V,h)$ its Lie algebra.**

Proof. Step 1:

$$0 = \frac{d}{dt}h(e^{tA}v, e^{tA}w) = h(e^{tA}A(v), e^{tA}w) + h(e^{tA}v, e^{tA}A(w)).$$

If h is e^{tA}-invariant, this gives $0 = h(Av,w) + h(v,Aw)$, hence A is antisymmetric.

Step 2: Conversely, suppose that A is antisymmetric. Then

$$\frac{d}{dt}h(e^{tA}v, e^{tA}w) = h(Ae^{tA}v, e^{tA}w) + h(e^{tA}v, Ae^{tA}Aw) = 0,$$

hence $h(e^{tA}v, e^{tA}w)$ is independent from t and equal to $h(v,w)$.

Step 3: It remains to show that any $g \in SO(V)$ in a connected component of this group satisfies $g = e^A$, where e^{tA} is orthogonal for all t; please do that as an exercise. ∎