Algebra and Geometry

lecture 9: positive self-adjoint operators

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December 8, 2016

Lagrange interpolation polynomials

THEOREM: Let k be a field, $x_1, ..., x_n \in k$ pairwise non-equal and $a_1, ..., a_n \in k$ any. **Then there exists a unique polynomial $P(t) \in k[t]$ of degree $\leqslant n-1$ such that $P(x_i) = a_i$.**

Proof. Step 1: Uniqueness: if $P_1(t), P_2(t)$ are two such polynomials, $P_1(t) - P_2(t)$ is a degree $\leqslant n-1$ polynomial which has at lleast n roots. By Bézout theorem, this is possible only if $P_1(t) - P_2(t) = 0$.

Step 2: Let

$$L_k(t) := \frac{\prod_{i \neq k}(t - x_i)}{\prod_{i \neq k}(x_k - x_i)}.$$

Then $L_k(x_i) = \delta_{ik}$, which gives $P(x_i) = a_i$ for $P(t) = \sum a_i L_i(t)$. ■

DEFINITION: The polynomial $P(t)$ constructed above is called **Lagrange interpolation polynomial**.

Adjoint operators

CLAIM: Let V be a vector space, g a scalar product on V, and $A \in \operatorname{End}(V)$. **Then there exists a unique operator $A^* \in \operatorname{End}(V)$ such that $g(A(x), y) = g(x, A^*(y))$ for all $x, y \in V$.**

Proof: Let $x_1, ..., x_n$ be an orthonormal basis in V, $A = (a_{ij})$ the matrix of A, and and A^t the transposed matrix $A^t = (a_{ji})$. Then $g(A(x_i), x_j) = a_{ij}$ and $g(x_i, A^*(x_j)) = a_i j$. This gives existence. Uniqueness is clear, because if $g(x, (A_1^* - A_2^*)(y)) = 0$ for all x, y, we have $A_1^* - A_2^* = 0$ **(prove it).** ■

DEFINITION: In this situation, the operator A^* is called **adjoint to A**. In orthonormal basis, **this operator is represented by the transposed matrix.**

CLAIM: $A \in O(V) \Leftrightarrow A^* = A^{-1}$.

Proof: The equality

$$g(A(x), A(y)) = g(x, y) \qquad (a)$$

holds for all x, y if and only if

$$g(x, A(y)) = g(A^{-1}(x), y). \qquad (b)$$

Indeed, (b) is obtained from (a) by $g(A^{-1}(x), y) = g(AA^{-1}(x), A(y)) = g(x, A(y))$, and (a) from (b) by $g(A(x), A(y)) = g(A^{-1}A(x), y) = g(x, y)$. ■

Self-adjoint operators

DEFINITION: Let V be a vector space and $g \in \mathrm{Sym}^2 V$ a scalar product. An operator $A: V \longrightarrow V$ is called **self-adjoint** if $A = A^*$.

REMARK: In orthonormal basis, a self-adjoint operator is given by a matrix that satisfies $A = A^t$, that is, **symmetric**, The self-adjoint operators are often called **symmetric operators**.

CLAIM: Let A be a self-adjoint operator on (V, g), and $g_A(x, y) := g(A(x), y)$. **Then g_A is a bilinear symmetric form on V.** Moreover, **the map $A \mapsto g_A$ gives a bijective correspondence between self-adjoint operators and bilinear symmetric forms on V.**

Proof: Using g to identify V and V^*, we obtain that the spaces $V^* \otimes V^*$ of bilinear symmetric forms and $\mathrm{End}(V) = V \otimes V^*$ are also identified. This identification is given by a map $A \mapsto g(A(\cdot), \cdot)$. By definition, the form $g_A(\cdot, \cdot) := g(A(\cdot), \cdot)$. is symmetric if and only if A is self-adjoint. ■

REMARK: This is just another way to construct the well-known **bijective correspondence between symmetric matrices and bilinear symmetric forms.**

Positive operators

DEFINITION: A self-adjoint operator A is called **positive** if $g(A(x), x) \geqslant 0$ for any $x \in V$.

REMARK: Positive self-adjoint operators **are identified with positive bilinear symmetric forms.**

In Lecture 8 we proved the following theorem

THEOREM: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

In the language of positive operators it can be stated as follows.

Theorem 1: (spectral theorem for self-adjoint operators)
Let A be a self-adjoint operator on (V, g). **Then A can be diagonalized in an orthonormal basis.**

Square root of a positive operator

DEFINITION: Let $A, B \in \operatorname{End} V$. We say that B **is a square root of** A if $B^2 = A$.

REMARK: As in the case of numbers, **square root is not unique**. However, for a connected Lie group G, **each element $\varphi \in G$ can be obtained as $\varphi = e^A$, which gives $(e^{\frac{A}{2}})^2 = \varphi$.** We proved this result for orthogonal group in Lecture 8.

EXERCISE: Find a matrix $A \in SL(3, \mathbb{R})$ which has infinitely many square roots.

THEOREM: Let A be a positive self-adjoint operator on (V, g). **Then there exists a unique positive self-adjoint $V \in \operatorname{End}(V)$ such that $B^2 = A$,** denoted by $B = \sqrt{A}$. Moreover, **the map $A \longrightarrow \sqrt{A}$ is continuous.**

Proof. Step 1: (existence):
Let $\alpha_1, ..., \alpha_n$ be the set of all eigenvalues of A. By Theorem 1, A is diagonal in a certain orthonormal basis, and all eigenvalues of A are real and non-negative. Consider the Lagrange interpolation polynomial $Q_A(t)$ mapping each α_i to $\sqrt{\alpha_i}$. Then $Q_A(A)$ is diagonal in the same orthonormal basis and satisfies $Q_A(A)^2 = A$.

Square root of a positive operator (2)

THEOREM: Let A be a positive self-adjoint operator on (V, g). **Then there exists a unique positive self-adjoint $V \in \operatorname{End}(V)$ such that $B^2 = A$,** denoted by $B = \sqrt{A}$. Moreover, **the map $A \longrightarrow \sqrt{A}$ is continuous.**

We have shown that $Q_A(A)^2 = A$ for a polynomial $Q_A(t) \in \mathbb{R}[t]$.

Step 2: Let B be a positive operator satisfying $B^2 = A$. **Since $BA = B^3 = AB$, B and A commute.** Therefore, **B commutes with $Q_A(A)$.**

Step 3: Consider the orthogonal decomposition $V = V_0 \oplus V_0^\perp$, where $V_0 = \ker A$. This decompostion is A-invariant by Theorem 1. Since B is positive, it is diagonalizable, hence $\operatorname{im} B^2 = \operatorname{im} B = \operatorname{im} A = V_0^\perp$, and $\ker B = V_0$. It remains to show that $Q_A(A) = B$ on $V_0^\perp$. Therefore, **we may assume that A is invertible.**

Step 4: Since B commutes with $Q_A(A)$, we have

$$Q_A(A)^2 - B^2 = (Q_A(A) - B)(Q_A(A) + B) = 0.$$

Since $Q_A(A)$ and B are positive definite, **the sum $Q_A(A)+B$ is also positive definite, and therefore invertible.** This implies that $Q_A(A) - B = 0$.

It remains only to show that the map $A \mapsto Q_A(A)$ is continuous.

Square root of a positive operator (3)

We use the following lemma.

LEMMA: Let $f: X \longrightarrow Y$ be a bijective continuous map of Hausdorff topological spaces with X compact. **Then f is a homeomorphism.**

Proof: A subset of X is closed if and only if it is compact. However, an image of a compact set under continuous map is compact **(prove it).** Therefore, an image of a closed set is closed, and inverse map f^{-1} is continuous. ■

Square root of a positive operator (4)

THEOREM: Let A be a positive self-adjoint operator on (V,g). **Then there exists a unique positive self-adjoint $V \in \operatorname{End}(V)$ such that $B^2 = A$,** denoted by $B = \sqrt{A}$. Moreover, **the map $A \mapsto \sqrt{A}$ is continuous.**

Step 5 (continuity of $A \mapsto \sqrt{A}$):
Let (V,g) be a vector space equipped with a positive definite scalar product, and S_r be the set of all positive adjoint operators A with all eigenvalues bounded by r. This is a closed subset of set of matrices obtained as $O(V) \cdot V_r$, where V_r is the set of all diagonal matrices with eigenvalues $0 \leqslant \alpha_i \leqslant r$, and $O(V)$ acts on V_r as $g(T) = gTg^{-1}$. **Since V_r and $O(V)$ are compact, S_r is compact.**

Step 6: The map $A \mapsto A^2 : S_r \longrightarrow S_{r^2}$ is bijective (as shown in Step 4) and clearly continuous, hence it is a homeomorphism. ■

Signature of a bilinear form

LEMMA: The set of all positive definite bilinear symmetric forms is connected.

Proof: Let g, g' be positive definite. Then g is connected to g' be the interval $tg + (1-t)g'$, where $t \in [0,1]$. ■

DEFINITION: Let h be a non-degenerate bilinear symmetric form on $V = \mathbb{R}^n$, and $x_1, ..., x_n$ an orthogonal basis. **The signature** of h is the number of positive and negative numbers in the set $\{h(x_1,x_1), h(x_2,x_2), ..., h(x_n,x_n)\}$.

THEOREM: The signature of a form h is independent from the choice of the basis $x_1, ..., x_n$.

Proof. Step 1: Let g be an auxiliary positive definite form on V, and A_h the self-adjoint operator associated with h as above. Since A_h is diagonal in appropriate orthogonal basis, A_h^2 is positive. Denote by B the matrix $\sqrt{A_h^2}$. Then AB^{-1} is a diagonal matrix with eigenvalues ± 1. **Signature of A is $(p,q) \Leftrightarrow \operatorname{Tr}(AB^{-1}) = p - q$.**

Step 2: It remains to show that the number $\operatorname{Tr}(AB^{-1})$ is independent from the choice of g. However, this number is by construction continuous in g and integral. **Since the set of positive definite g is connected, this function is constant.** ■