Algebra and Geometry

lecture 10: polar decomposition and ruled hyperboloids

Misha Verbitsky

Université Libre de Bruxelles

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Self-adjoint operators (reminder)

DEFINITION: Let V be a vector space and $g \in \operatorname{Sym}^2 V$ a scalar product. An operator $A: V \longrightarrow V$ is called **self-adjoint** if $A = A^*$.

REMARK: In orthonormal basis, a self-adjoint operator is given by a matrix that satisfies $A = A^t$, that is, **symmetric**, The self-adjoint operators are often called **symmetric operators**.

CLAIM: Let A be a self-adjoint operator on (V, g), and $g_A(x, y) := g(A(x), y)$. **Then g_A is a bilinear symmetric form on V.** Moreover, **the map $A \mapsto g_A$ gives a bijective correspondence between self-adjoint operators and bilinear symmetric forms on V.**

Proof: Using g to identify V and V^*, we obtain that the spaces $V^* \otimes V^*$ of bilinear symmetric forms and $\operatorname{End}(V) = V \otimes V^*$ are also identified. This identification is given by a map $A \mapsto g(A(\cdot), \cdot)$. By definition, the form $g_A(\cdot, \cdot) := g(A(\cdot), \cdot)$. is symmetric if and only if A is self-adjoint. ■

REMARK: This is just another way to construct the well-known **bijective correspondence between symmetric matrices and bilinear symmetric forms.**

Positive operators (reminder)

DEFINITION: A self-adjoint operator A is called **positive** if $g(A(x), x) \geqslant 0$ for any $x \in V$.

REMARK: Positive self-adjoint operators **are identified with positive bilinear symmetric forms.**

In Lecture 8 we proved the following theorem

THEOREM: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

In the language of positive operators it can be stated as follows.

THEOREM: (spectral theorem for self-adjoint operators)
Let A be a self-adjoint operator on (V, g). **Then A can be diagonalized in an orthonormal basis.**

We also proved

THEOREM: Let A be a possitive adjoint operator. **Then there exists a unique positive, adjoint B such that $B^2 = A$.**

Square root of a positive operator (reminder)

DEFINITION: Let $A, B \in \operatorname{End} V$. We say that B **is a square root of** A if $B^2 = A$.

REMARK: As in the case of numbers, **square root is not unique**. However, for a connected Lie group G, **each element** $\varphi \in G$ **can be obtained as** $\varphi = e^A$, **which gives** $(e^{\frac{A}{2}})^2 = \varphi$. We proved this result for orthogonal group in Lecture 8.

EXERCISE: Find a matrix $A \in SL(3, \mathbb{R})$ which has infinitely many square roots.

THEOREM: Let A be a positive self-adjoint operator on (V, g). **Then there exists a unique positive self-adjoint** $V \in \operatorname{End}(V)$ **such that** $B^2 = A$, denoted by $B = \sqrt{A}$. Moreover, **the map** $A \longrightarrow \sqrt{A}$ **is continuous.**

Proof. Step 1: (existence):
Let $\alpha_1, ..., \alpha_n$ be the set of all eigenvalues of A. By Theorem 1, A is diagonal in a certain orthonormal basis, and all eigenvalues of A are real and non-negative. Consider the Lagrange interpolation polynomial $Q_A(t)$ mapping each α_i to $\sqrt{\alpha_i}$. Then $Q_A(A)$ is diagonal in the same orthonormal basis and satisfies $Q_A(A)^2 = A$.

Square root of a positive operator (2)

THEOREM: Let A be a positive self-adjoint operator on (V,g). **Then there exists a unique positive self-adjoint $V \in \operatorname{End}(V)$ such that $B^2 = A$,** denoted by $B = \sqrt{A}$. Moreover, **the map $A \longrightarrow \sqrt{A}$ is continuous.**

We have shown that $Q_A(A)^2 = A$ for a polynomial $Q_A(t) \in \mathbb{R}[t]$.

Step 2: Let B be a positive operator satisfying $B^2 = A$. **Since $BA = B^3 = AB$, B and A commute.** Therefore, **B commutes with $Q_A(A)$.**

Step 3: Consider the orthogonal decomposition $V = V_0 \oplus V_0^\perp$, where $V_0 = \ker A$. This decompostion is A-invariant by Theorem 1. Since B is positive, it is diagonalizable, hence $\operatorname{im} B^2 = \operatorname{im} B = \operatorname{im} A = V_0^\perp$, and $\ker B = V_0$. It remains to show that $Q_A(A) = B$ on $V_0^\perp$. Therefore, **we may assume that A is invertible.**

Step 4: Since B commutes with $Q_A(A)$, we have

$$Q_A(A)^2 - B^2 = (Q_A(A) - B)(Q_A(A) + B) = 0.$$

Since $Q_A(A)$ and B are positive definite, **the sum $Q_A(A)+B$ is also positive definite, and therefore invertible.** This implies that $Q_A(A) - B = 0$.

It remains only to show that the map $A \mapsto Q_A(A)$ is continuous.

Square root of a positive operator (3)

We use the following lemma.

LEMMA: Let $f: X \longrightarrow Y$ be a bijective continuous map of Hausdorff topological spaces with X compact. **Then f is a homeomorphism.**

Proof: A subset of X is closed if and only if it is compact. However, an image of a compact set under continuous map is compact. Indeed, (metrizable) compacts can be chacterized as topological spaces where all sequences have converging subsequences, but a continuous image of such a set always have the same property.

Therefore, an image of a closed set is closed, and inverse map f^{-1} is continuous. ■

Square root of a positive operator (4)

THEOREM: Let A be a positive self-adjoint operator on (V, g). **Then there exists a unique positive self-adjoint $V \in \operatorname{End}(V)$ such that $B^2 = A$,** denoted by $B = \sqrt{A}$. Moreover, **the map $A \mapsto \sqrt{A}$ is continuous.**

Step 5 (continuity of $A \mapsto \sqrt{A}$):
Let (V, g) be a vector space equipped with a positive definite scalar product, and S_r be the set of all positive adjoint operators A with all eigenvalues bounded by r. This is a closed subset of set of matrices obtained as $O(V) \cdot V_r$, where V_r is the set of all diagonal matrices with eigenvalues $0 \leqslant \alpha_i \leqslant r$, and $O(V)$ acts on V_r as $g(T) = gTg^{-1}$. **Since V_r and $O(V)$ are compact, S_r is compact.**

Step 6: The map $A \mapsto A^2 : S_r \longrightarrow S_{r^2}$ is bijective (as shown in Step 4) and clearly continuous, hence it is a homeomorphism. ■

Polar decomposition

THEOREM: Let V be a vector space with scalar product over $\mathbb{R}$, and $A \in GL(V)$ a invertible matrix. **Then $A = PU$, where U is orthogonal and P positive and adjoint,** and this decomposition is unique.

The proof is later today.

REMARK: The operators U and P are not necessarily commuting.

REMARK: This decomposition is continuous in A (by construction).

EXERCISE: A couple of lectures ago it was shown that $SO(n)$ is connected. **Deduce from polar decomposition that the space $SL(n, \mathbb{R})$ is also connected.**

Action on bilinear symmetric forms

LEMMA: Let V be a vector space with scalar product g, and $R \in GL(V)$ a invertible operator. Consider the bilinear symmetric form $R(g)(x,y) := g(R(x), R(y))$. Let A be the corresponding positive adjoint operator, $g(A(x),y) := R(g)$, and $\sqrt{A}$ its square root, and $B = (\sqrt{A})^{-1}$. **Then BR is orthogonal.**

Proof:

$$RB(g)(x,y)) = g(R(B(x)), R(B(y))) = \\ g(AB(x), B(y)) = g(AB(x), B(y)) = g(B^2A(x), y) = g(x,y).$$

■

Same argument also brings the following corollary.

COROLLARY: Let $R \in GL(V)$ a invertible operator, and R a positive self-adjoint operator. **Then BR is orthogonal if and only if $B^2 = A^{-1}$.** ■

THEOREM: Let V be a vector space with scalar product over $\mathbb{R}$, and $R \in GL(V)$ a invertible matrix. **Then $R = UP$, where U is orthogonal and P positive and adjoint,** and this decomposition is unique.

Proof: Let $R(g)(x,y) = g(A(x),y)$ be the corresponding positive self-adjoint map, and $B = (\sqrt{A})^{-1}$. Then $U := BR$ is orthogonal, hence $R = B^{-1}U$. ■

Quadratic forms on $\mathbb{R}^2$

DEFINITION: Let q be a quadratic form on a vector space V. A vector $v \in V$ is called **isotropic** if $q(v) = 0$.

Proposition 1: Let $Q(xe_1 + ye_2) = ax^2 + by^2 + 2cxy$ be a non-degenerate quadratic form on $\mathbb{R}^2$. **Then the set $\{v \in \mathbb{R}^2 \ \ | \ \ Q(v) = 0\}$ if isotropic vectors is either a union of two lines intersecting in 0, or $\{0\}$ depending on signature.**

Proof: If Q is positive definite or negative definite, it is $\{0\}$. If the signature is $(1,1)$, let $u, v \in \mathbb{R}^2$ be the basis such that the corresponding bilinear symmetric form satisfies $q(u,u) = 1, q(v,v) = -1, q(u,v) = 0$. Then the vectors $w_+ := \frac{u+v}{2}$ and $w_- := \frac{u-v}{2}$ are isotropic and satisfy $q(w_+, w_-) = 1$. No linear combination of form $aw_+ + bw_-$, with $a, b \neq 0$ can be isotropic, because $q(aw_+ + bw_-) = a^2 q(w_+, w_+) + b^2 q(w_-, w_-) + 2ab q(w_-, w_+) = 2ab$. ■

Affine quadrics

DEFINITION: Let Q be a quadratic form on V and λ a linear form. Then the set $S := \{v \in V \ | \ Q(v) + \lambda(v) + c = 0\}$ is called **affine quadric**.

REMARK: **Intersection of an affine quadric and an affine subspace $W \subset V$ is an affine quadric in W.**

DEFINITION: Let $\tilde{Q}(v) := Q(v) + \lambda(v) + c$, $S := \{v \in V \ | \ \tilde{Q}(v) = 0\}$ be an affine quadric, $W \subset V$ an affine hyperplane inersecting S in $s \in S$, and $W_0 := \{x - y \ | \ x, y \in W\}$ the corresponding linear hyperplane. We say that **W is tangent to S in s** if for each $l \in W_0$, the function $l \longrightarrow \tilde{Q}(s - l)$ is quadratic on l.

Remark 1: Let W be tangent to an affine quadric in s. Choose an affine coordinate system on V with $0 = s$. **Then W is a linear hyperplane, and $\tilde{Q}(v)$ is a quadratic form on W.**

Affine quadrics in $\mathbb{R}^3$

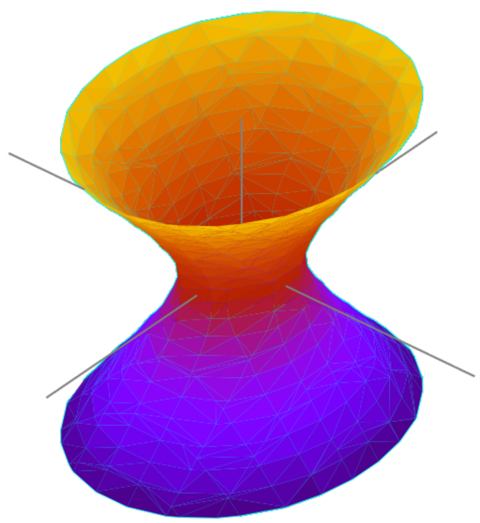
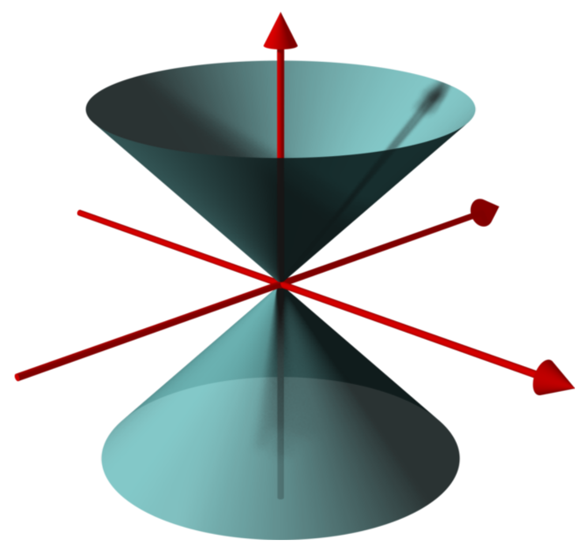
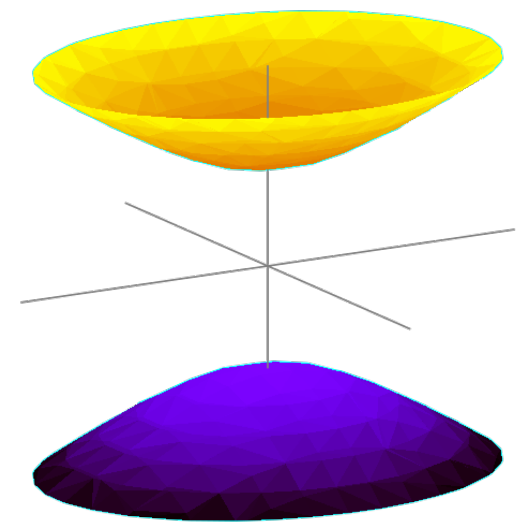
PROPOSITION: Let Q be a non-degenerate quadratic form on $V := \mathbb{R}^3$, $c \neq 0$, and $S := \{v \in V \mid Q(v) = c\}$ an affine quadric. Consider a tangent hyperplane $W \subset V$. **Then the quadratic part of $Q|_W$ is non-degenerate.**

Proof. Step 1: Let $W_0 := \{x - y \mid x, y \in W\}$ be linearization of the tangent hyperplane, and s the tangent point. Then W_0 is given by $\lambda(v) = 0$, where λ is linear part of $v \longrightarrow Q(s + v)$, that is, $\lambda(v) = q(s, v)$, where q is the corresponding bilinear symmetric form. **We obtain that $W_0 = s^\perp$.**

Step 2: Since $Q(s) \neq 0$, the space $W_0^\perp = \langle s \rangle$ does not intersect W_0, and $Q|_{W_0}$ is non-degenerate. ■

Hyperboloids

DEFINITION: Let Q be a quadratic form on $\mathbb{R}^3$ of signature (2,1), and $c \neq 0$. The affine quadric $S := \{v \in V \mid Q(v) = c\}$ is called **hyperboloid**. When $c > 0$, it is called **hyperbolic**, or **one-sheeted hyperboloid**, or **ruled hyperboloid** and when $c < 0$, it is **elliptic**, or **two-sheeted hyperboloid**

CLAIM: Let $S := \{v \in V \mid Q(v) = c\}$ be a hyperboloid, and W a tangent plane. **Then $Q|_W$ is positive definite for $c < 0$ and has signature $(1,1)$ for $c > 0$.**

Proof: The corresponding linearization W_0 is $s^\perp$, when $c = Q(s) > 0$ it has signature (1,1), when $c = Q(s) > 0$ it has signature (2,0). ■

Ruled hyperboloid

PROPOSITION: Let Q be a quadratic form on $\mathbb{R}^3$ of signature (2,1), $c > 0$. and $S := \{v \in V \ \mid \ Q(v) = c\}$ the corresponding hyperboloid. **Then for any tangent plane W, the intersection $W \cap S$ is union of two lines.**

Proof: Let s be the tangent point, and choose an affine coordinate system such that $s = 0$. Then the tangent plane W is linear, and $Q(v) - c$ is a quadratic form on W of signature (1,1) (Remark 1). **Then $W \cap S$ is a union of two lines by Proposition 1.** ■

DEFINITION: A 2-dimensional surface $S \subset \mathbb{R}^3$ is called **ruled** if each point of S is contained on a line $l \subset S$.

Quadrics of rotation

DEFINITION: Fix a positive definite form g on $\mathbb{R}^3$, and let Q be a non-degenerate quadratic form on $\mathbb{R}^3$. In appropriate orthonormal coordinates, Q can be written as

$$Q(x, y, z) = \pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2}. \quad (*)$$

The coordinate axes of this coordinate system are called **axes of the quadric** S. **They are defined uniquely up to an automorphism of $\mathbb{R}^3$ preserving g and Q.**

DEFINITION: We say that a quadric $S := \{v \in V \quad | \quad Q(v) = c\}$ **has rotational symmetry** if it is preserved by an isometric rotation of $\mathbb{R}^3$.

CLAIM: A quadric has rotational symmetry if and only if two of the coefficients in (*) are equal.

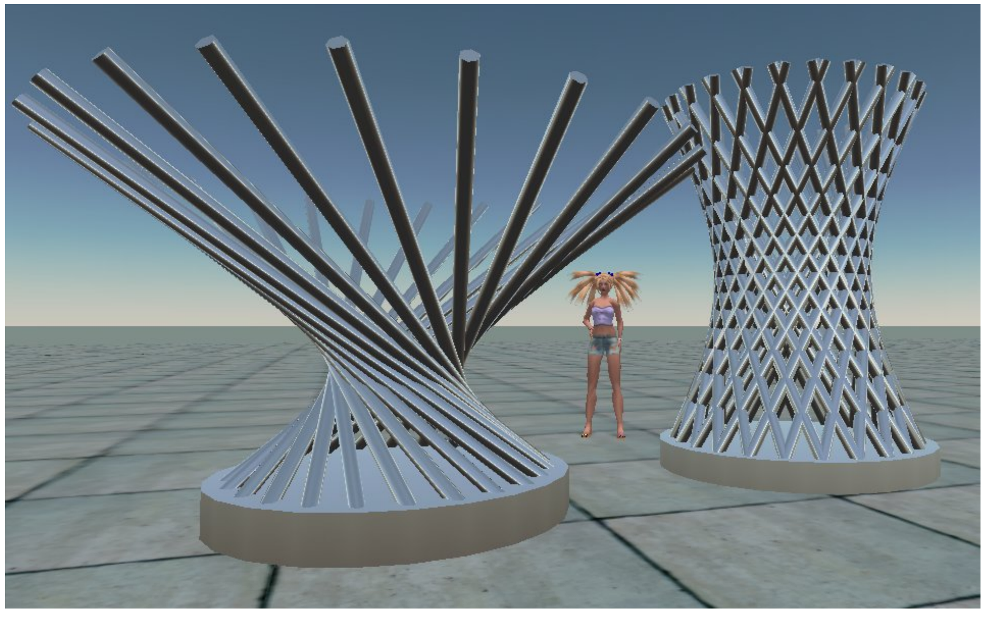
Proof: Clearly, if (say) two of the coefficients in (*) are equal and $Q(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{a^2} \pm \frac{z^2}{c^2}$, then rotation around an axis $x = y = 0$ preserves Q.

Conversely, when all coefficients in (*) are different, the orthonormal basis (x, y, z) is determined uniquely up to a sign, and any isometry preserving Q also preserves these three axes. ■

Hyperboloid of rotation

PROPOSITION: Let $l_1, l_2 \subset \mathbb{R}^3$ be two non-perpendicular skew lines (that is, likes which are not parallel and don't intersect), and S a surface of rotation obtained by rotating l_1 around l_2. **Then S is a hyperboloid of rotation.** Conversely, **any ruled hyperboloid of rotation is obtained this way**.

Proof: Let S be a hyperboloid of rotation. Then S contains a line. Since it is rotationally symmetric, it can be obtained by rotating this line around the central axis.



Conversely, any two non-perpendicular skew lines can be related by affine transform commuting with rotation around the first line **(prove it)**, and an affine transform maps quadrics to quadrics. ■