MATH-F-303, test assignment 4 (tensor product)

Rules: This is a class assignment for discussion.

Definition 4.1. Symmetric group Σ_n acts on $V^{\otimes n} := \underbrace{V \otimes V \otimes ... \otimes V}_{n \text{ times}}$ exchanging the components. Denote by $\mathsf{Sym}^n V$ the Σ_n-invariant part of $V^{\otimes n}$.

Exercise 4.1. Prove that $(\mathsf{Sym}^2 V)^*$ is the space of bilinear symmetric forms on V.

Exercise 4.2. Let V be n-dimensional. Find dimension of $\mathsf{Sym}^3 V$.

Exercise 4.3. Let $\Lambda^n V$ be the space of tensors $\psi \in V^{\otimes n}$ which are antisymmetric under exchange of any two components. Find dimension of $\Lambda^n V$ if V is k-dimensional.

Exercise 4.4. The space of Cartan tensors $C(V)$ is the space of tensors $\psi \in V \otimes V \otimes V$ which are symmetric under exchange first two components and antisymmetric under exchange of the last two. Prove that $V \otimes V \otimes V = \mathsf{Sym}^3 V \oplus \Lambda^3 V \oplus C(V)$.

Exercise 4.5. Let $\psi \in C(V) \subset V \otimes V \otimes V$ be a tensor that is invariant under the cyclic permutation of the 3 tensor components. Prove that $\psi = 0$.

Definition 4.2. Let g be a non-degenerate bilinear symmetric form on a vector space V, and $A \in \operatorname{End} V$ an endomorphism. We say that A is **antisymmetric** if $g(A(x), y) = -g(x, A(y))$ for all $x, y \in V$. Denote the space of antisymmetric tensors by $\mathfrak{so}(V)$.

Exercise 4.6. Consider a non-degenerate bilinear symmetric form g as an isomorphism $g\ V \longrightarrow V^*$, and let $\mathsf{Id} \otimes g^{-1}: \ V \otimes V^* \longrightarrow V \otimes V$ be the induced isomorphism. Prove that $\mathsf{Id} \otimes g^{-1}(\mathfrak{so}(V)) = \Lambda^2 V$.

Exercise 4.7 (*). Let V be a vector space, and g a bilinear symmetric form. A linear map $A: \ V \longrightarrow V$ is called **orthogonal** if $g(x, y) = g(A(x), A(y))$. Suppose that V is a vector space over $\mathbb{C}$. Find an orthogonal endomorphism A which cannot be diagonalized (that is, there is no basis where A is diaginal).

Exercise 4.8. Let $A: \ V \longrightarrow V$ be an endomorphism of finite order: $A^n = \mathsf{Id}_V$, where V is a vector space over $\mathbb{C}$. Prove that A can be diagonalized.