MATH-F-303, test assignment 8 (Grassmann algebra)

Rules: Please solve this in class and put the solutions to my mailbox in the mailroom on 9-th floor (near the library). You have 2 hours (from 2 to 4 PM) to wite your solutions. Please sign the papers. Your results contribute towards your final grade.

Definition 8.1. Let $\alpha \in \Lambda^2 V^*$ be a skew-symmetric 2-form on V. We say that α is **symplectic** if for any $v \in V$, there exists $w \in V$ such that $\alpha(v, w) \neq 0$.

Remark 8.1. In lectures we have seen that for each symplectic form on V there exists a special basis in V such that this form has a standard matrix in this basis. If you don't recall this, please consult your notes.

Exercise 8.1. Let $\omega \in \Lambda^2 V^*$, $V = \mathbb{R}^{2n}$, $n \geqslant 2$ be a symplectic form, and $\alpha \in \Lambda^1 V^*$ a non-zero 1-form. Prove that $\alpha \wedge \omega \neq 0$.

Exercise 8.2. Let $\omega_1, \omega_2 \in \Lambda^2(\mathbb{R}^4)$ symplectic forms, satisfying $\omega_1 \wedge \omega_2 = 0$. Prove that $(\omega_1 + \omega_2) \wedge (\omega_2 - \omega_1) = 0$, or find a counterexample.

Exercise 8.3. Let $V = \mathbb{R}^{2n}$, and $U \subset \Lambda^2 V^*$ be the set of all symplectic forms. Prove that it is open in the sense of the usual topology on $\Lambda^2 V^* = \mathbb{R}^{\frac{n(n-1)}{2}}$.

Exercise 8.4. Let $V = \mathbb{R}^4$, and $\alpha \in \Lambda^2 V$ be a non-zero antisymmetric 2-tensor. Prove that there exists $\beta \in \Lambda^2 V$ such that $\alpha \wedge \beta \neq 0$.

Exercise 8.5. Let $V = \mathbb{R}^4$, and $\alpha \in \Lambda^2 V^*$. Assume that $\alpha \wedge \alpha \in \Lambda^4 V^*$ is non-degenerate. Prove that α is a symplectic 2-form.