MATH-F-303, test assignment 9 (trace and determinant)

Rules: Please solve this in class and give solutions to Roberto. Please sign the papers. Your results contribute towards your final grade. Each problem of this set is worth 2 usual problems.

Definition 6.1. **Exponent** of an endomorphism A is $\sum_{n=0}^{\infty} \frac{A^n}{n!}$. **Trace** of A represented by the matrix (a_{ij}) is $\sum_i a_{ii}$.

Exercise 6.1. Let k be a vector space. Consider the bilinear map $V \times V^* \longrightarrow k$ mapping (x, λ) to $\lambda(x)$, and let $\Psi: \ V \otimes V^* \longrightarrow k$ be the corresponding map on tensor product. Consider the isomorphism $\operatorname{End}(V) = V \otimes V^*$ obtained in lectures. Prove that $\operatorname{Tr}(A) = \Psi(A)$ for any $A \in \operatorname{End}(V) = V \otimes V^*$.

Remark 6.1. This implies that the definition of trace is independent of the basis.

Exercise 6.2. Prove that $\det e^A = e^{\operatorname{Tr} A}$.

Exercise 6.3. Let $A \in \operatorname{End}(V)$, $\dim V = n$, and let $P(t) = t^n + a_{n-1} t^{n-1} + ... + a_0$ be its characteristic polynomial. Extend the action of A to the Grassmann algebra $\Lambda^* V$ by multiplicativity: $A(x_1 \wedge x_2 \wedge ... \wedge x_k) = A(x_1) \wedge A(x_2) \wedge ... \wedge A(x_k)$. Prove that the trace of A on $\Lambda^k V$ is equal to a_{n-k}.