MATH-F-420: final exam

Rules: Every student receives from me a list of 8 exercises (chosen randomly), and has to solve 4 of them by January 17. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solution, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes). Maximal score is 4 of 8 exercises, but every exercise you solve brings you closer. Feel free to google the solutions if you are able. Problems with "k points" are worth k usual exercises.

1 Manifolds and rings of smooth functions

Exercise 1.1. Consider **the Moebius strip** M as a quotient space of $\mathbb{R} \times [0,1]$ with opposite lines glued together with reverse orientation. Construct a submanifold in $\mathbb{R}^3$ which is diffeomorphic to a Moebius strip, or prove that it does not exist.

Exercise 1.2. Let M be an n-dimensional manifold. Construct a smooth, surjective map from M to the torus $(S^1)^n$.

Exercise 1.3. Let R be a ring of continuous $\mathbb{R}$-valued functions on a compact topological space M, and $I \subset R$ an ideal. Prove that there exists $Z \subset M$ such that I is the ideal of all functions vanishing at Z, or find a counterexample.

Exercise 1.4 (3 points). Let M and M' be smooth manifolds, such that the ring $C^\infty M$ is isomorphic to $C^\infty M'$ over the field of real numbers. Prove that M is diffeomorphic to M'.

Exercise 1.5. Let R be a be a ring of germs of continuous functions on $\mathbb{R}^n$ in 0, and $K \subset R$ – intersection of all powers of maximal ideal. Prove that K is generated by all non-negative $\phi \in K$, or find a counterexample.

Exercise 1.6. Let $X, Y \subset M$ be closed subsets of a metric space M. Assume that $\inf_{x\in X, y\in Y} d(x,y) > 1$. Prove that there exists a 1-Lipschitz function on M which is equal to 1 on X and 0 on Y.

Exercise 1.7. Let M be a manifold, f a continuous function on M, and F the set of all smooth functions on M satisfying $g \leqslant F$. Prove $f(x) = \sup_{g\in F} g(x)$.

Exercise 1.8. Let M be a connected, compact topological manifold, and G a group of all its homeomorphisms. Prove that G acts on M transitively.

Exercise 1.9. Suppose that the group $G = \mathbb{Z}^2$ act on $\mathbb{R}$ by diffeomorphisms in such a way that any $x \in G, x \neq 1$ acts non-trivially. Prove that the quotient $\mathbb{R}/G$ is not a manifold.

2 Sheaves

Definition 2.1. A sheaf $\mathcal{B}$ is called **flasque** if any restriction map $\mathcal{B}(U) \longrightarrow \mathcal{B}(V)$ is surjective.

Exercise 2.1. For a given sheaf $\mathcal{B}_1$, find a sheaf monomorphism $\mathcal{B}_1 \hookrightarrow \mathcal{B}$ to a flasque sheaf.

Definition 2.2. A sheaf $\mathcal{I}$ is called **injective** if for any sheaf morphism $\mathcal{B} \xrightarrow{\phi} \mathcal{I}$ and a monomorphism $\mathcal{B} \hookrightarrow \mathcal{B}'$, the map ϕ can be extended to a morphism $\mathcal{B}' \xrightarrow{\phi} \mathcal{I}$.

Exercise 2.2. Prove that any injective sheaf is flasque.

Exercise 2.3. Let B be a sheaf of modules over $C^\infty M$, and $\check{B}$ a sheaf given by $\check{B}(U) = B_c(U)^*$, where $B_c(U)$ is the space of sections with compact support. Prove that $\check{B}(U)$ is a flasque sheaf.

Definition 2.3. A sheaf $\mathcal{F}$ on a topological space M is called **soft** if for any closed subset $X \subset M$, the natural map from the space of global sections $\mathcal{F}(M)$ to the space $\mathcal{F}(X)$ of germs of $\mathcal{F}$ in X is surjective.

Exercise 2.4. Let M be a smooth compact manifold, and B a sheaf of modules over $C^\infty M$. Prove that B is soft.

Definition 2.4. A function is called **meromorphic** if it is expressed as a quotient of two holomorphic (that is, complex analytic) functions: $\mu = \frac{f}{g}$. It is defined on the complement of the set where $g = 0$.

Exercise 2.5. Let $M = \mathbb{C}$, $\mathcal{O}_M$ be the sheaf of holomorphic functions on M, and $\mathcal{M}(M)$ the sheaf of meromorphic functions. Prove that the quotient sheaf $\mathcal{M}(M)/\mathcal{O}_M$ is flasque.

Exercise 2.6. Let Ψ be a morphism of ringed spaces mapping $(\mathbb{R}^n, C^{i+1})$ to $(\mathbb{R}^n, C^i)$. Prove that Ψ maps $\mathbb{R}^n$ to a point.

3 Vector bundles

All vector bundles here are considered over reals (and not over complex numbers).

Exercise 3.1. Let $TS^2 \oplus \mathbb{R}$ be a direct sum of a tangent bundle TS^2 and a trivial 1-dimensional bundle. Is the bundle $TS^2 \oplus \mathbb{R}$ trivial?

Exercise 3.2. Let B be a non-trivial vector bundle on a smooth manifold M, and $B_1 := B \oplus C^\infty M$ a direct sum of B and a trivial 1-dimensional bundle. Prove that B is non-trivial, or find a counterexample.

Exercise 3.3. Let M be a smooth manifold, and TM the total space of its tangent bundle. Prove that TM is orientable, or find a counterexample.

Exercise 3.4. Let M be an oriented manifold. Prove that all bundles $\Lambda^i M$ are oriented, or find a counterexample.

Exercise 3.5. Let M be a simply connected manifold. Prove that any real rank 1 bundle on M is trivial.

Exercise 3.6. Let B be a vector bundle. Prove that $B \otimes B$ is oriented, or find a counterexample.

Exercise 3.7 (2 points). Let B be a vector bundle. Prove that $\Lambda^2 B$ is oriented, or find a counterexample.

Exercise 3.8. Prove that all real vector bundles on $\mathbb{R}$ are trivial. Construct a non-trivial vector bundle on S^1 or prove it does not exist.

Exercise 3.9 (2 points). Construct a rank 2 vector bundle not admitting a non-degenerate bilinear form of signature (1,1), or prove that such bundle does not exist.

Exercise 3.10. Let $M_1 \xrightarrow{\phi} M$ be a surjective, smooth map without critical points, M, M_1 compact manifolds, and B a non-trivial bundle on M. Can the pullback bundle $\phi^* B$ be trivial?

Exercise 3.11. Let B be an oriented vector bundle of rank 2. Prove that B admits a non-degenerate skew-symmetric form.

Exercise 3.12. Construct a non-trivial rank 2 vector bundle which does not have any non-trivial sub-bundles.

4 De Rham algebra

Exercise 4.1. Let ω be a non-degenerate skew-symmetric 2-form (such a form is called "symplectic") on a $2n$-dimensional vector space V and g a positive definite scalar product. Prove that there exists an orthonormal basis $x_1, ..., x_{2n}$ in V such that $\omega = \sum_{i=1}^n \alpha_i y_{2i-1} \wedge y_{2i}$, where y_i is the dual basis in V^*.

Exercise 4.2. Let $\eta \in \Lambda^k V$ be a non-zero form, and $L_\eta : \ \Lambda^1 \longrightarrow \Lambda^{1+k} V$ the multiplication map $x \longrightarrow x \wedge \eta$. Prove that its kernel $\ker L_\eta$ is at most k-dimensional. For $k = n-2$, and $n = \dim V$ even, prove that $\ker L_\eta$ cannot be 1-dimensional, or find a counterexample.

Exercise 4.3. Let $\delta : \ \Lambda^* M \longrightarrow \Lambda^* M$ be a derivation of de Rham algebra, and α a differential form with support in a closed set $S \subset M$. Prove that $\delta(\alpha)$ has support in S.

Exercise 4.4. Let α be a closed differential form with compact support on $\mathbb{R}^n$. Prove that there exists a k-form β with support in a unit ball such that $\alpha - \beta$ is exact.

Exercise 4.5. Let $\tau : \ \Lambda^*(M) \longrightarrow \Lambda^{*-1}(M)$ be a derivation shifting grading by -1. Prove that there exists a vector field $v \in TM$ such that $\tau = i_v$ is the contraction with this vector field, $\tau(\alpha) = \alpha(v, \cdot, \cdot, ...)$.

Exercise 4.6. Let θ be an exact 1-form, and $d_\theta(x) = dx + \theta \wedge x$. Prove that cohomology of d_θ are isomorphic to de Rham cohomology.

Exercise 4.7. Find a 1-form η on $\mathbb{R}^7$ such that $\eta \wedge (d\eta)^3$ is nowhere zero.