Geometry of manifolds

lecture 3

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Rings and derivations

REMARK: All rings in these lectures are assumed to be commutative and with unit. Algebras are associative, but not necessarily commutative (such as the matrix algebra). **Rings over a field k** are rings containing a field k. We assume that k has characteristic 0.

DEFINITION: Let A be a ring over a field k. A k-linear map $D\ A \longrightarrow A$ is called **a derivation** if it satisfies **the Leibnitz identity** $D(fg) = D(f)g + gD(f)$. The space of derivations is denoted as $\operatorname{Der}_k(A)$.

EXAMPLE: $\frac{d}{dt}: \mathbb{C}[t] \longrightarrow \mathbb{C}[t]$. $\frac{d}{dt}: C^\infty\mathbb{R} \longrightarrow C^\infty\mathbb{R}$.

REMARK: Any derivation $\delta \in \operatorname{Der}_k(A)$ vanishes on $k \subset A$. Indeed, $\delta(1) = \delta(1 \cdot 1) = 2\delta(1)$.

CLAIM: Let K be **a finite extension** of a field k, that is, a field containing k and finite-dimensional as a K-linear space. **Then $\operatorname{Der}_k(K) = 0$.**

Proof: Indeed, any $x \in K$ satisfies a non-trivial polynomial equation $P(x) = 0$ with coefficients in k. Chose $P(t)$ of smallest degree possible. **For any $\delta \in \operatorname{Der}_k(A)$, we have $0 = \delta(P(x)) = P'(x)\delta(x)$,** and unless $\delta(x) = 0$, one has $P'(x) = 0$, giving a contradiction. ■

Modules over a ring

DEFINITION: Let A be a ring over a field k. **An A-module** is a vector space V over k, equipped with an algebra homomorphism $A \longrightarrow \mathrm{End}(V)$, where $\mathrm{End}(V)$ denotes the endomorphism algebra of V, that is, the matrix algebra.

REMARK: Let A be a field. Then A-modules are the same as vector spaces over A.

DEFINITION: Homomorphisms, isomorphisms, submodules, quotient modules, direct sums of modules are defined in the same way as for the vector spaces. A ring A is itself an A-module. A direct sum of n copies of A is denoted A^n. Such A-module is called **a free A-module**.

EXAMPLE: A-submodules in A are the same as ideals in A.

DEFINITION: Finitely generated A-module is a quotient module of A^n.

Derivations as an A-module

REMARK: Let A be a ring over k. **The space $\operatorname{Der}_k(A)$ of derivations is also an A-module,** with multiplicative action of A given by $rD(f) = rD(f)$.

CLAIM: Let $A = k[t_1, .., t_k]$ be a polynomial ring. **Then $\operatorname{Der}_k(A)$ is a free A-module isomorphic to A^n,** with generators $\frac{d}{dt_1}, \frac{d}{dt_2}, ..., \frac{d}{dt_n}$.

Proof: Consider a map $\operatorname{Der}_k(A) \longrightarrow A^n$,

$$D \longrightarrow (D(t_1), D(t_2), ..., D(t_n))$$

It is surjective, because it maps each $\frac{d}{dt_i}$ to $(0, ..., 0, 1, 0, ..., 0)$, and injective, because each derivation which vanishes on t_i, vanishes on the whole polynomial ring. ■

Now we prove a similar result for $C^\infty \mathbb{R}^n$.

Hadamard's Lemma

LEMMA: (Hadamard's Lemma)
Let f be a smooth function on $\mathbb{R}^n$, and x_i the coordinate functions. **Then $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, for some smooth $g_i \in C^\infty \mathbb{R}^n$.**

Proof: Let $t \in \mathbb{R}^n$. Consider a function $h(t) \in C^\infty \mathbb{R}^n$, $h(t) = f(tx)$. Using chain rule, we get $\frac{dh}{dt} = \sum \frac{d}{dx_i} f(tx) x_i$, obtaining

$$f(x) - f(0) = \int_0^1 \frac{dh}{dt} dt = \sum_i x_i \int_0^1 \frac{df(tx)}{dx_i}(tx) dt.$$

■

COROLLARY: Let $\mathfrak{m}_0$ be an ideal of all smooth functions on $\mathbb{R}^n$ vanishing in 0. **Then $\mathfrak{m}_0$ is generated by coordinate functions.** ■

COROLLARY: Let f be a smooth function on $\mathbb{R}^n$ satisfying $f(x) = 0$ and $f'(x) = 0$. **Then $f \in \mathfrak{m}_x^2$.**

Proof: $f(x) = \sum_{i=1}^n x_i g_i(x)$, where all g_i vanish in 0. ■

Derivations of $C^\infty\mathbb{R}^n$

THEOREM: Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$, $A = C^\infty\mathbb{R}^n$, and $\operatorname{Der}(A) \xrightarrow{\Psi} (C^\infty\mathbb{R}^n)^n$ map D to $(D(x_1), D(x_2), ..., D(x_n))$. **Then $\Psi: \operatorname{Der}(C^\infty\mathbb{R}^n) \longrightarrow A^n$ is an isomorphism.**

Proof. Step 1: Since Ψ maps each $\frac{d}{dt_i}$ to $(0, ..., 0, 1, 0, ..., 0)$, it is **surjective.**

Step 2: Let $\mathfrak{m}_0$ be an ideal of functions vanishing in 0, and $\delta \in \ker\Psi$. Then $\Psi(x_i) = 0$, where x_i are coordinate functions. By Hadamard's Lemma, $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, hence $\delta(f) = \sum_{i=1}^n x_i \delta(g_i)$. **Therefore, $\delta(f)$ lies in $\mathfrak{m}_0$.**

Step 3: Same argument proves that $\delta(f)$ vanishes everywhere, for all $f \in C^\infty M$. ■

Sheaves

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below is a more abstract version of the notion of "sheaf of functions" defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and exact sequences

DEFINITION: A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

CLAIM: A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta\big|_{U_i \cap U_j}$ and $-\eta\big|_{U_j \cap U_i}$.

Ringed spaces (reminder)

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A sheaf of functions is a subsheaf in the sheaf of all functions, closed under multiplication.

For simplicity, I assume now that a sheaf of rings is a subsheaf in the sheaf of all functions.

DEFINITION: A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Smooth manifolds (reminder)

DEFINITION: Let $(M,\mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M,\mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n,\mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: **Diffeomorphism** of smooth manifolds is a homeomorphism φ which induces an isomorphisms of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Partition of unity (reminder)

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

DEFINITION: Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

DEFINITION: Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$ on M. A point $x \in M$ does not lie in the **support** $\operatorname{Sup}(f)$ of f if $f|_U = 0$ for some neighbourhood $U \ni x$. A section is called **section with compact support** or **supported on a compact set** if its support is compact.

REMARK: **Support of a section is obviously closed.**

Vector fields as derivations

DEFINITION: Let M be a smooth manifold. A **vector field** on M is an element in $\operatorname{Der}(C^\infty M)$.

Pros of this definition: it is entirely coordinate-free.
Cons: Restriction to an open subset is a complicated business.

EXAMPLE: For $M = \mathbb{R}^n$, **the space $\operatorname{Der}(C^\infty M)$ is a free module generated by $\frac{d}{dx_i}$, $i = 1, ..., n$.**

REMARK: We want to prove that vector fields form a sheaf. **However, it is not immediately clear how to restrict a vector field from U to $W \subset U$.**

Idea: Let $V \subset U$. To show that $\operatorname{Der}(M)$ is a sheaf we have to learn how to differentiate functions in V using vectors $X \in \operatorname{Der}(U)$. When f has compact support in V, it is extended to a smooth function on U tautologically, and you can differentiate. **It remais to show that the space $\operatorname{Der}_c(V)$ of derivations on functions with compact support in V is equal to $\operatorname{Der}(V)$.**