Geometry of manifolds

lecture 4

Misha Verbitsky

Université Libre de Bruxelles

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Modules over a ring (reminder)

DEFINITION: Let A be a ring over a field k. **An A-module** is a vector space V over k, equipped with an algebra homomorphism $A \longrightarrow \operatorname{End}(V)$, where $\operatorname{End}(V)$ denotes the endomorphism algebra of V, that is, the matrix algebra.

EXAMPLE: A ring A is itself an A-module. A direct sum of n copies of A is denoted A^n. Such A-module is called **a free A-module**.

EXAMPLE: **A-submodules in A are the same as ideals in A.**

Rings and derivations (reminder)

REMARK: All rings in these lectures are assumed to be commutative and with unit. Algebras are associative, but not necessarily commutative (such as the matrix algebra). **Rings over a field k** are rings containing a field k. We assume that k has characteristic 0.

DEFINITION: Let A be a ring over a field k. A k-linear map $D\ A \longrightarrow A$ is called **a derivation** if it satisfies **the Leibnitz identity** $D(fg) = D(f)g + gD(f)$. The space of derivations is denoted as $\operatorname{Der}_k(A)$.

REMARK: Let A be a ring over k. **The space $\operatorname{Der}_k(A)$ of derivations is also an A-module,** with multiplicative action of A given by $rD(f) = rD(f)$.

THEOREM: Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$, $A = C^\infty\mathbb{R}^n$, and $\operatorname{Der}(A) \xrightarrow{\Psi} (C^\infty\mathbb{R}^n)^n$ map D to $(D(x_1), D(x_2), ..., D(x_n))$. **Then $\Psi : \operatorname{Der}(C^\infty\mathbb{R}^n) \longrightarrow A^n$ is an isomorphism.**

Sheaves (reminder)

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below is a more abstract version of the notion of "sheaf of functions" defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Ringed spaces (reminder)

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A sheaf of functions is a subsheaf in the sheaf of all functions, closed under multiplication.

For simplicity, I assume now that a sheaf of rings is a subsheaf in the sheaf of all functions.

DEFINITION: A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Smooth manifolds (reminder)

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: Diffeomorphism of smooth manifolds is a homeomorphism φ which induces an isomorphisms of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Partition of unity (reminder)

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

DEFINITION: Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

DEFINITION: Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$ on M. A point $x \in M$ does not lie in the **support** $\operatorname{Sup}(f)$ of f if $f|_U = 0$ for some neighbourhood $U \ni x$. A section is called **section with compact support** or **supported on a compact set** if its support is compact.

REMARK: **Support of a section is obviously closed.**

Vector fields as derivations

DEFINITION: Let M be a smooth manifold. A **vector field** on M is an element in $\operatorname{Der}(C^\infty M)$.

EXAMPLE: For $M = \mathbb{R}^n$, **the space $\operatorname{Der}(C^\infty M)$ is a free module generated by $\frac{d}{dx_i}$, $i = 1, ..., n$.**

Pros of this definition: it is entirely coordinate-free.
Cons: Restriction to an open subset is a complicated business.

THEOREM: Let $U \Subset V$ be open subset of a smooth metrizable manifold, and $D \in (C^\infty M)$ a derivation. Consider a smooth function $\Phi_{U,V} \in C^\infty M$ supported on V, and equal to 1 on U. Given $f \in C^\infty V$, define $D(f)|_U := D(\Phi_{U,V} f)$. Choosing a cover $\{U_i\}$ of such sets, we can glue together a section $D(f)$ of $C^\infty V$ from such $D(f)\Big|_{U_i}$. **This operation is independent of all choices we made and gives an element $D|_V \in \operatorname{Der}(V)$. Moreover, this restriction maps define a structure of a sheaf on $\operatorname{Der}(M)$.**

Proof: Later in this lecture.

Direct limits

DEFINITION: Commutative diagram of vector spaces is given by the following data. There is a directed graph (graph with arrows). For each vertex of this graph we have a vector space, and each arrow corresponds to a homomorphism of the associated vector spaces. **These homomorphism are compatible, in the following way.** Whenever there exist two ways of going from one vertex to another, the compositions of the corresponding arrows are equal.

DEFINITION: Let $\mathcal{C}$ be a commutative diagram of vector spaces, A, B – vector spaces, corresponding to two vertices of a diagram, and $a \in A, b \in B$ elements of these vector spaces. Write $a \sim b$ if a and b are mapped to the same element $d \in D$ by a composition of arrows from $\mathcal{C}$. Let $\sim$ be an equivalence relation generated by such $a \sim b$. A quotient $\bigoplus_i C_i/E$ is called **a direct limit** of a diagram $\{C_i\}$. The same notion is also called **colimit** and **inductive limit**. Direct limit is denoted $\varinjlim$.

DEFINITION: Let $\mathcal{F}$ be a sheaf on M, $x \in M$ a point, and $\{U_i\}$ the set of all neighbourhoods of x. Consider a diagram with the set of vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. The **space of germs** of $\mathcal{F}$ in x is a direct limit $\varinjlim \mathcal{F}(U_i)$ over this diagram. The space of germs is also called **the stalk** of the sheaf $\mathcal{F}$.

Germs of functions

DEFINITION: A diagram $\mathcal{C}$ is called **filtered** if for any two vertices C_i, C_j, there exists a third vertex C_k, and sequences of arrows leading from C_i to C_k and from C_j to C_k.

EXAMPLE: The diagram formed by all neighbourhoods of a point is obviously filtered.

CLAIM: Let $\mathcal{C}$ be a commutative diagram of vector spaces C_i, with all C_i equipped with a ring structure, and all arrows ring homomorphisms. Suppose that the diagram $\mathcal{C}$ is filtered. **Then there exists a unique ring structure on $C := \varinjlim C_i$ such that all the maps $C_i \longrightarrow C$ are ring homomorphisms.**

DEFINITION: Let $M, \mathcal{F}$ be a ringed space, $x \in M$ its point, and $\{U_i\}$ the set of all its neighbourhoods. Consider a commutative diagram with vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. For each vertex U_i we take a vector space of sections $\mathcal{F}(U_i)$, and for each arrow the corresponding restriction map. The direct limit of this diagram is called **the ring of germs of the sheaf $\mathcal{F}$ in x**.

Morphisms of sheaves

DEFINITION: Let $\mathcal{B}, \mathcal{B}'$ be sheaves on M. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \downarrow & & \downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

DEFINITION: A sheaf morphism is called **injective**, or **a monomorphism** if it is injective on stalks and **surjective**, or **epimorphism** if it is surjective on stalks.

EXERCISE: Let $\mathcal{B} \xrightarrow{\varphi} \mathcal{B}'$ be an injective morphism of sheaves on M. **Prove that φ induces an injective map $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$ for each U.**

REMARK: A sheaf epimorphism $\mathcal{B} \xrightarrow{\varphi} \mathcal{B}'$ **does not necessarily induce a surjective map $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$.**

DEFINITION: **A sheaf isomorphism** is a homomorphism $\Psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi: \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$.

EXERCISE: Show that a morphism of sheaves $\Psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ **is an isomorphism if and only if it is epi and mono.**

Sheaves of modules

REMARK: Let $A : \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\varphi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: **A vector bundle** on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Dual sheaves

CLAIM: Let $U \subset V$ be open subsets of a Hausdorff space M. **A section $f \in \mathcal{F}(U)$ with compact support $Z \subset U$ can be uniquely extended to $\tilde{f} \in \mathcal{F}(V)$, also with support in Z.**

Proof: Consider a cover $\{U_1 = U, U_2 = V \backslash Z\}$ of V, and let $f_1 = f \in F(U_1)$ and $f_2 = 0 \in F(U_2)$. Since $f_i\big|_{U_1 \cap U_2} = 0$, we can glue f_1 and f_2, obtaining the extension $\tilde{f}$ and $0 \in \mathcal{F}(U_2)$. ■

DEFINITION: Let $\mathcal{F}$ be a sheaf. Denote the space of sections of $\mathcal{F}$ on U with compact support by $\mathcal{F}_c(U)$. Let $\mathcal{F}^*(U)$ map U to the dual space $\mathcal{F}_c(U)^*$. Using the claim above, we obtain a restriction map $\mathcal{F}^*(V) \longrightarrow \mathcal{F}^*(U)$ for each open $V \supset U$. This gives **dual presheaf** $\mathcal{F}^*$

EXERCISE: Let M be a manifold, and $\mathcal{F}$ a sheaf of modules over $C^\infty M$. **Prove that $\mathcal{F}^*$ is a sheaf.**

HINT: Use partition of unity.

Smooth functions with prescribed support

EXERCISE: Let $X, Y \subset M$ be non-intersecting closed subsets in a metric space. **Find non-intersecting open neighbourhoods $U_1 \supset X$ and $U_2 \supset U$.**

Proposition 1: Let $U \Subset V$ – open subsets in a smooth metrizable manifold. **Then there exists a smooth function $\Phi_{U,V} \in C^\infty M$, supported on V, and equal to 1 on U.**

Proof. Step 1: Let U_1, U_2 be non-intersecting open sets containing the closure $X = \overline{U}$ and $Y = M \backslash V$, and $U_3 = V \backslash \overline{U}$. Since $U_1 \cup U_2$ contains $\overline{U}$ and $M \backslash V$, **U_1, U_2, U_3 is a cover of M.**

Step 2: Consider a cover of M by open sets $\{V_i\}$ which are contained in either U_1, U_2 or U_3, but never intersect both U_1 and U_2. Let ψ_i be a partition of unity supported in V_i, and $\Phi_{U,V}$ be the sum of all ψ_i with support in $U_1 \cap U_3$ and intersecting U_2. Then $\Phi_{U,V} = 0$ in $M \backslash U_2 \supset M \backslash V$, because support of $\Phi_{U,V}$ does not intersect U_2, and $\Phi_{U,V} = 1$ on $U_1 \supset U$, because $\Phi_{U,V}$ is a sum of all ψ_i with support intersecting U_2. ■

Local operators

DEFINITION: A linear map $\Psi : C^\infty(M) \longrightarrow C^\infty(M)$ is called **local** if for any function f supported in a compact subset $Z \subset M$, its image $\Psi(f)$ is supported in Z.

LEMMA: Any derivation $D: C^\infty(M) \longrightarrow C^\infty(M)$ is local.

Proof: Let f be a function supported in Z. For each g with support outside of Z, we have $0 = D(fg) = fD(g) + gD(f)$.

Proposition 1 gives a function g which is equal to 1 on any compact subset K not intersecting Z and 0 in a neighbourhood of Z. Then $0 = D(fg)|_K = gD(f)|_K = D(f)$. **Therefore, K does not intersect support $D(f)$.** However, the union of all such K is $M\backslash Z$. ■

REMARK: By definition, **differential operator** is an operator expressed through derivations and multiplication by a function. From the above lemma, we obtain that **all differential operators are local**. The converse is also true, but harder to prove: **any local operator on $C^\infty M$ is a differential operator**.

Local operators and sheaves

CLAIM: For any local operator $D: C^\infty M \longrightarrow C^\infty M$, and any $f \in C^\infty M$, **the germ $D(f)$ in x is determined uniquely by the germ of f in x.**

Proof: Consider $f, f_1 \in C^\infty M$ with the same germ in x. Then $f - f_1 = 0$ in a neighbourhood $U \ni x$, hence $D(f) = D(f_1)$ in this neighbourhood (by locality of D). ■

Proposition 2: Let $D: C^\infty M \longrightarrow C^\infty M$ be a local operator, $U \subset M$ an open subset, and $f \in C^\infty U$ a function with germs f_x at each $x \in U$. **Then there exists a unique function $D(f) \in C^\infty U$ such that its germs are equal to $D(f_x)$,** where $D(f_x)$ is an application of D to the germ f_x defined as above.

Proof. Step 1: Consider a partition of unity $\sum \psi_i = 1$ on U, and let $D(f) = \sum_i D(\psi_i f)$. Since the functions $\psi_i f$ have compact support, they can be extended to M, and $\sum_i D(\psi_i f)$ is well defined.

Step 2: For any $x \in U$, let g be the sum of $\psi_i f$ with support of ψ_i containing x. Then $\sum_i D(\psi_i f)_x = D(g)_x$, because the rest of summands have support outside of x. **However, the germs of these functions are equal: $g_x = f_x$. This implies that the germs of $\sum_i D(\psi_i f)$ are equal to $D(f_x)$.** ■

Derivations as a sheaf

PROPOSITION: Let $U \subset V$ be open subsets of a smooth manifold M, and $D \in \operatorname{Der}(C^\infty V)$. Define $D|_U$ as a derivation with the same germs at each point as D (Proposition 2). **This defines a structure of a sheaf $U \longrightarrow \operatorname{Der}(U)$.**

Proof. Step 1: A vector field is uniquely determined by its restriction to the germs of all sections, hence a derivation D which vanishes on all germs for all $x \in M$ vanishes everywhere. **This takes care of the first sheaf axiom.**

Proof. Step 2: Let $\{U_i\}$ be a cover of M. To glue a derivation D from its bits $D_i \in \operatorname{Der}(C^\infty(U_i))$, consider a partition of unity ψ_i subordinate to $\{U_i\}$. **Then $D(f) := \sum D_i(\psi_i f)$ is a derivation which restricts to all D_i.** ∎

COROLLARY: The sheaf of derivations is locally free, that is, $\operatorname{Der} C^\infty M$ defines a vector bundle on M.

DEFINITION: It is called **the tangent bundle**, and denoted TM.