Geometry of manifolds

lecture 5

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Sheaves (reminder)

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below is a more abstract version of the notion of "sheaf of functions" defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Ringed spaces (reminder)

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A sheaf of functions is a subsheaf in the sheaf of all functions, closed under multiplication.

For simplicity, I assume now that a sheaf of rings is a subsheaf in the sheaf of all functions.

DEFINITION: A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Smooth manifolds (reminder)

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: Diffeomorphism of smooth manifolds is a homeomorphism φ which induces an isomorphisms of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Partition of unity (reminder)

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

DEFINITION: Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

DEFINITION: Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$ on M. A point $x \in M$ does not lie in the **support** $\operatorname{Sup}(f)$ of f if $f|_U = 0$ for some neighbourhood $U \ni x$. A section is called **section with compact support** or **supported on a compact set** if its support is compact.

REMARK: Support of a section is obviously closed.

Sheaves of modules (reminder)

REMARK: Let $A: \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\varphi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: Locally free sheaf of modules over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: A vector bundle on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Germs of sheaves (reminder)

DEFINITION: Let M be a manifold, $\mathcal{F}$ a sheaf, and $x \in M$ a point, and U_1, U_2 its neighbourhoods. For any two sections $\gamma_1 \in F(U_1)$, $\gamma_2 \in F(U_2)$, we write $\gamma_1 \sim \gamma_2$ if $\gamma_1|_U = \gamma_2|_U$ for some neighbourhood $U \ni x$ contained in $U_1 \cap U_2$. The space of equivalence classes is called **stalk of a sheaf F in x,** or **the space of germs**, and its elemens are called **germs of F in x**.

REMARK: If $\mathcal{O}$ is a sheaf of rings, **the space of its germs in x is a ring.** If F is a sheaf of modules over a sheaf of rings $\mathcal{O}$, **the space of germs F_x of F in x is a $\mathcal{O}_x$-module.**

DEFINITION: A morphism of sheaves is **injective** (or **a monomorphism**) if it is injective on stalks, and **surjective** (or **an epimorphism**) if it is surjective on stalks.

Sections with compact support (reminder)

CLAIM: Let $U \subset V$ be open subsets of a Hausdorff space M. **A section $f \in \mathcal{F}(U)$ with compact support $Z \subset U$ can be uniquely extended to $\tilde{f} \in \mathcal{F}(V)$, also with support in Z.**

Proof: Consider a cover $\{U_1 = U, U_2 = V \backslash Z\}$ of V, and let $f_1 = f \in F(U_1)$ and $f_2 = 0 \in F(U_2)$. Since $f_i\big|_{U_1 \cap U_2} = 0$, we can glue f_1 and f_2, obtaining the extension $\tilde{f}$. ■

EXERCISE: Let $X, Y \subset M$ be non-intersecting closed subsets in a metric space. **Find non-intersecting open neighbourhoods $U_1 \supset X$ and $U_2 \supset U$.**

PROPOSITION: Let $U \Subset V$ – open subsets in a smooth metrizable manifold. **Then there exists a smooth function $\Phi_{U,V} \in C^\infty M$, supported on V, and equal to 1 on U.**

Local operators (reminder)

DEFINITION: A linear map $\Psi: C^\infty(M) \longrightarrow C^\infty(M)$ is called **local** if for any function f supported in a compact subset $Z \subset M$, its image $\Psi(f)$ is supported in Z.

LEMMA: Any derivation $D: C^\infty(M) \longrightarrow C^\infty(M)$ is local.

REMARK: By definition, **differential operator** is an operator expressed through derivations and multiplication by a function. From the above lemma, we obtain that **all differential operators are local**. The converse is also true, but harder to prove: **any local operator on $C^\infty M$ is a differential operator**.

Local operators and sheaves (reminder)

CLAIM: For any local operator $D: C^\infty M \longrightarrow C^\infty M$, and any $f \in C^\infty M$, **the germ $D(f)$ in x is determined uniquely by the germ of f in x.**

Proof: Consider $f, f_1 \in C^\infty M$ with the same germ in x. Then $f - f_1 = 0$ in a neighbourhood $U \ni x$, hence $D(f) = D(f_1)$ in this neighbourhood (by locality of D). ■

PROPOSITION: Let $D: C^\infty M \longrightarrow C^\infty M$ be a local operator, $U \subset M$ an open subset, and $f \in C^\infty U$ a function with germs f_x at each $x \in U$. **Then there exists a unique function $D(f) \in C^\infty U$ such that its germs are equal to $D(f_x)$,** where $D(f_x)$ is an application of D to the germ f_x defined as above.

Proof. Step 1: Consider a partition of unity $\sum \psi_i = 1$ on U, and let $D(f) = \sum_i D(\psi_i f)$. Since the functions $\psi_i f$ have compact support, they can be extended to M, and $\sum_i D(\psi_i f)$ is well defined.

Step 2: For any $x \in U$, let g be the sum of $\psi_i f$ with support of ψ_i containing x. Then $\sum_i D(\psi_i f)_x = D(g)_x$, because the rest of summands have support outside of x. **However, the germs of these functions are equal: $g_x = f_x$. This implies that the germs of $\sum_i D(\psi_i f)$ are equal to $D(f_x)$.** ■

Derivations as a sheaf (reminder)

PROPOSITION: Let $U \subset V$ be open subsets of a smooth manifold M, and $D \in \operatorname{Der}(C^\infty V)$. Define $D|_U$ as a derivation with the same germs at each point as D (Proposition 2). **This defines a structure of a sheaf $U \longrightarrow \operatorname{Der}(U)$.**

Proof. Step 1: A vector field is uniquely determined by its restriction to the germs of all sections, hence a derivation D which vanishes on all germs for all $x \in M$ vanishes everywhere. **This takes care of the first sheaf axiom.**

Proof. Step 2: Let $\{U_i\}$ be a cover of M. To glue a derivation D from its bits $D_i \in \operatorname{Der}(C^\infty(U_i))$, consider a partition of unity ψ_i subordinate to $\{U_i\}$. **Then $D(f) := \sum D_i(\psi_i f)$ is a derivation which restricts to all D_i.** ■

COROLLARY: The sheaf of derivations is locally free, that is, $\operatorname{Der} C^\infty M$ defines a vector bundle on M.

DEFINITION: It is called **the tangent bundle**, and denoted TM.

Locally trivial fibrations

DEFINITION: A smooth map $f: X \longrightarrow Y$ is called **a locally trivial fibration** if each point $y \in Y$ has a neighbourhood $U \ni y$ such that $f^{-1}(U)$ is diffeomorphic to $U \times F$, and the map $f: f^{-1}(U) = U \times F \longrightarrow U$ is a projection. In such situation, F is called **the fiber** of a locally trivial fibration.

DEFINITION: **A trivial fibration** is a map $X \times Y \longrightarrow Y$.

EXAMPLE: The projection $S^3 \subset \mathbb{C}^2 \backslash 0 \xrightarrow{f} \mathbb{C}P^1$ is called **the Hopf fibration**. Given $U = \{x : 1\} \subset \mathbb{C}P^1$, with $|x| \leqslant 1$, one has

$$f^{-1}(U) = \{z_1, z_2 \in S^3 \ \mid \ |z_1|^2 + |z_2|^2 = 1, |z_1| \leqslant 1\}$$

(here z_i are complex coordinates in $\mathbb{C}^2$). Then

$$f^{-1}(U) = \left\{(z_1, z_2) \ \mid \ z_2 \in U(1) \cdot \sqrt{1 - |z_1|^2}\right\},$$

where $U(1) = \{z \in \mathbb{C} \ \mid \ |z| = 1\}$. Therefore, **the Hopf fibration $f: S^3 \longrightarrow S^2$ is a locally trivial fibration**.

REMARK: Since $\pi_1(S^3) = 0$ and $\pi_1(S^1 \times S^2) = \mathbb{Z}$, **the Hopf fibration is non-trivial.**

Vector bundles

DEFINITION: A vector bundle on Y is a locally trivial fibration $f: X \longrightarrow Y$ with fiber $\mathbb{R}^n$, with each fiber $V := f^{-1}(y)$ equipped with a structure of a vector space, smoothly depending on $y \in Y$.

REMARK: This definition **is not very precise or rigorous,** because "smoothly depending on $y \in Y$" **needs to be explained.**

REMARK: This definition is compatible with the one we used previously ("a vector bundle is a locally free sheaf of $C^\infty M$-modules"). This will be explained later.

For a more rigorous approach:

1. Define categories.

2. Define group objects and vector space objects

3. Formulate "smoothly depending on $y \in Y$" in these terms.

Categories: data

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: The set $\mathcal{O}$b$(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}$b$(\mathcal{C})$, one has a set $\mathcal{M}$or(X, Y) of **morphisms from X to Y**.

Composition of morphisms: For each $\varphi \in \mathcal{M}$or$(X, Y), \psi \in \mathcal{M}or(Y, Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{M}or(X, Z)$

Identity morphism: For each $A \in \mathcal{O}$b$(\mathcal{C})$ there exists a morphism Id$_A \in \mathcal{M}$or(A, A).

REMARK: In some versions of axiomatic set theory, one considers not a set, but **a class** of objects, which could be arbitrarily big, such as the class of all sets, or the class of all linear spaces. The category with **a set** of morphisms and objects is called **a small category**, and one with a class **a big category**.

Categories: axioms

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{M}\text{or}(X,Y)$, one has $\operatorname{Id}_x \circ \varphi = \varphi = \varphi \circ \operatorname{Id}_Y$

DEFINITION: Let $X, Y \in \mathcal{O}\text{b}(\mathcal{C})$ – objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}\text{or}(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}\text{or}(Y,X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:

Category of sets: its morphisms are arbitrary maps.

Category of vector spaces: its morphisms are linear maps.

Categories of rings, groups, fields: morphisms are homomorphisms.

Category of topological spaces: morphisms are continuous maps.

Category of smooth manifolds: morphisms are smooth maps.

It is often convenient to express morphisms by arrows, and call them "maps".

Some categorical constructions

DEFINITION: A product $X_1 \times X_2$ of $X_1, X_2 \in \mathcal{O}\mathrm{b}(\mathcal{C})$ is an object of $\mathcal{C}$ equipped with **projection maps** $\pi_i : X_1 \times X_2 \longrightarrow X_i$ such that **for any pair of morphisms** $\varphi_i \in \mathcal{M}\mathrm{or}(Y, X_i)$ **there is a unique morphism** $\varphi \in \mathcal{M}\mathrm{or}(Y, X_1 \times X_2)$ **such that** $\varphi \circ \pi_i = \varphi_i$**.**

EXERCISE: Prove that **a product is unique up to isomorphism**.

EXERCISE: Prove that the product is **associative:** $X \times (Y \times Z) \cong (X \times Y) \times Z$ and **commutative:** $X \times Y \cong Y \times X$.

EXERCISE: Find the product in the categories of a. groups b. rings c. vector spaces d. sets e. topological spaces.

More categorical constructions

DEFINITION: An initial object of a category is an object $I \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(I, X)$ is always a set of one element. **A terminal object** is $T \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(X, T)$ is always a set of one element.

EXERCISE: Prove that **the initial and the terminal object is unique,** up to isomorphism.

EXERCISE: Find the initial and the terminal object in the categories of a. groups b. rings c. vector spaces d. sets e. topological spaces, or show that it does nor exist.

EXERCISE: Let T be a terminal object. **Prove that $X \times T \cong X$ for each $X \in \mathcal{O}b(\mathcal{C})$.**