Geometry of manifolds

lecture 6

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Sheaves (reminder)

DEFINITION: **An open cover** of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below is a more abstract version of the notion of "sheaf of functions" defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Ringed spaces (reminder)

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A sheaf of functions is a subsheaf in the sheaf of all functions, closed under multiplication.

For simplicity, I assume now that a sheaf of rings is a subsheaf in the sheaf of all functions.

DEFINITION: A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Smooth manifolds (reminder)

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: **Diffeomorphism** of smooth manifolds is a homeomorphism φ which induces an isomorphisms of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Partition of unity (reminder)

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

DEFINITION: Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

DEFINITION: Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$ on M. A point $x \in M$ does not lie in the **support** $\operatorname{Sup}(f)$ of f if $f|_U = 0$ for some neighbourhood $U \ni x$. A section is called **section with compact support** or **supported on a compact set** if its support is compact.

REMARK: Support of a section is obviously closed.

Sheaves of modules (reminder)

REMARK: Let $A: \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\varphi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: **A vector bundle** on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Germs of sheaves (reminder)

DEFINITION: Let M be a manifold, $\mathcal{F}$ a sheaf, and $x \in M$ a point, and U_1, U_2 its neighbourhoods. For any two sections $\gamma_1 \in F(U_1)$, $\gamma_2 \in F(U_2)$, we write $\gamma_1 \sim \gamma_2$ if $\gamma_1|_U = \gamma_2|_U$ for some neighbourhood $U \ni x$ contained in $U_1 \cap U_2$. The space of equivalence classes is called **stalk of a sheaf F in x,** or **the space of germs**, and its elemens are called **germs of F in x**.

REMARK: If $\mathcal{O}$ is a sheaf of rings, **the space of its germs in x is a ring.** If F is a sheaf of modules over a sheaf of rings $\mathcal{O}$, **the space of germs F_x of F in x is a $\mathcal{O}_x$-module.**

DEFINITION: A morphism of sheaves is **injective** (or **a monomorphism**) if it is injective on stalks, and **surjective** (or **an epimorphism**) if it is surjective on stalks.

Locally trivial fibrations (reminder)

DEFINITION: A smooth map $f: X \longrightarrow Y$ is called **a locally trivial fibration** if each point $y \in Y$ has a neighbourhood $U \ni y$ such that $f^{-1}(U)$ is diffeomorphic to $U \times F$, and the map $f: f^{-1}(U) = U \times F \longrightarrow U$ is a projection. In such situation, F is called **the fiber** of a locally trivial fibration.

DEFINITION: **A trivial fibration** is a map $X \times Y \longrightarrow Y$.

DEFINITION: **A vector bundle** on Y is a locally trivial fibration $f: X \longrightarrow Y$ with fiber $\mathbb{R}^n$, with each fiber $V := f^{-1}(y)$ equipped with a structure of a vector space, smoothly depending on $y \in Y$.

REMARK: This definition **is not very precise or rigorous,** because "smoothly depending on $y \in Y$" **needs to be explained.**

REMARK: This definition is compatible with the one we used previously ("a vector bundle is a locally free sheaf of $C^\infty M$-modules"). This will be explained later.

Categories: data (reminder)

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: The set $\mathcal{O}$b$(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}$b$(\mathcal{C})$, one has a set $\mathcal{M}$or(X, Y) of **morphisms from X to Y**.

Composition of morphisms: For each $\varphi \in \mathcal{M}$or$(X, Y), \psi \in \mathcal{M}or(Y, Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{M}or(X, Z)$

Identity morphism: For each $A \in \mathcal{O}$b$(\mathcal{C})$ there exists a morphism $\mathrm{Id}_A \in \mathcal{M}or(A, A)$.

REMARK: In some versions of axiomatic set theory, one considers not a set, but **a class** of objects, which could be arbitrarily big, such as the class of all sets, or the class of all linear spaces. The category with **a set** of morphisms and objects is called **a small category**, and one with a class **a big category**.

Categories: axioms (reminder)

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{M}\mathrm{or}(X,Y)$, one has $\mathrm{Id}_x \circ \varphi = \varphi = \varphi \circ \mathrm{Id}_Y$

DEFINITION: Let $X, Y \in \mathcal{O}\mathrm{b}(\mathcal{C})$ – objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}\mathrm{or}(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}\mathrm{or}(Y,X)$ such that $\varphi \circ \psi = \mathrm{Id}_X$ and $\psi \circ \varphi = \mathrm{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:

Category of sets: its morphisms are arbitrary maps.

Category of vector spaces: its morphisms are linear maps.

Categories of rings, groups, fields: morphisms are homomorphisms.

Category of topological spaces: morphisms are continuous maps.

Category of smooth manifolds: morphisms are smooth maps.

It is often convenient to express morphisms by arrows, and call them "maps".

Some categorical constructions (reminder)

DEFINITION: A product $X_1 \times X_2$ of $X_1, X_2 \in \mathcal{O}b(\mathcal{C})$ is an object of $\mathcal{C}$ equipped with **projection maps** $\pi_i : X_1 \times X_2 \longrightarrow X_i$ such that **for any pair of morphisms** $\varphi_i \in \mathcal{M}or(Y, X_i)$ **there is a unique morphism** $\varphi \in \mathcal{M}or(Y, X_1 \times X_2)$ **such that** $\varphi \circ \pi_i = \varphi_i$**.**

EXERCISE: Prove that **a product is unique up to isomorphism**.

EXERCISE: Prove that the product is **associative:** $X \times (Y \times Z) \cong (X \times Y) \times Z$ and **commutative:** $X \times Y \cong Y \times X$.

EXERCISE: Find the product in the categories of a. groups b. rings c. vector spaces d. sets e. topological spaces.

More categorical constructions

DEFINITION: An initial object of a category is an object $I \in \mathcal{O}\mathrm{b}(\mathcal{C})$ such that $\mathcal{M}\mathrm{or}(I, X)$ is always a set of one element. **A terminal object** is $T \in \mathcal{O}\mathrm{b}(\mathcal{C})$ such that $\mathcal{M}\mathrm{or}(X, T)$ is always a set of one element.

EXERCISE: Prove that **the initial and the terminal object is unique,** up to isomorphism.

EXERCISE: Find the initial and the terminal object in the categories of a. groups b. rings c. vector spaces d. sets e. topological spaces, or show that it does nor exist.

EXERCISE: Let T be a terminal object. **Prove that $X \times T \cong X$ for each $X \in \mathcal{O}\mathrm{b}(\mathcal{C})$.**

Group objects in categories

DEFINITION: An object $G \in \mathcal{O}b(\mathcal{C})$ is called **a group object** if there exists a morphism $\mu \in \mathcal{M}or(G \times G, G)$ (**the product**), a morphism $e \in \mathcal{M}or(T, G)$ from the terminal object (**the unit**), and a morphism $i \in \mathcal{M}or(G, G)$ (**the inverse**), satisfying the following axioms.

Associativity: Consider the morphisms $\mu_{12}, \mu_{23} : G \times G \times G \longrightarrow G \times G$, the first map takes the product on the first two objects, and acts as identity on the third, the second maps is a product on last 2 and identity on the first. Then $\mu_{12} \circ \mu = \mu_{23} \circ \mu : G \times G \times G \longrightarrow G$.
Unit: The compositions $G = G \times T \xrightarrow{\operatorname{Id}_G \times e} G \times G \xrightarrow{\mu} G$ and $G = G \times T \xrightarrow{e \times \operatorname{Id}_G} G \times G \xrightarrow{\mu} G$ are identities.
Inverse: Let $\Delta : G \longrightarrow G \times G$ be **the diagonal map,** that is, a map $G \longrightarrow G \times G$ obtained from a pair of identity maps. Then the composition $G \xrightarrow{\Delta} G \times G \xrightarrow{\operatorname{Id}_G \times i} G \times G \xrightarrow{\mu} G$ is equal to $G \longrightarrow T \xrightarrow{e} G$.

EXAMPLE: A topological group is a group object in the category of topological spaces.

EXAMPLE: A Lie group is a group object in the category of smooth manifolds.

Topological groups over a base

DEFINITION: Fix a topological space M, and let $\mathcal{C}_M$ be a category of pairs $(X, f: X \longrightarrow M)$ with morphisms being continuous maps from X_1 to X_2 commuting with the projections to M. The product in $\mathcal{C}_M$ is called **fiber product:** $X_1 \times_M X_2 := \{(x_1, x_2) \in X_1 \times X_2 \ \mid \ f_1(x_1) = f_2(x_2)\}$. A group object in $\mathcal{C}_M$ is called **a topological group over M**.

REMARK: Let $\pi: G \longrightarrow M$ be a topological group over M. Then the fiber $\pi^{-1}(m)$ is a group for each $m \in M$. **This group structure depends on $m \in M$ continuously**, but **to state this dependency formaly, one needs to define a topological group over M.**

Topological groups over a base (without categories)

REMARK: This definition is equivalent to the following.

DEFINITION: Let $B \xrightarrow{\pi} M$ be a continuous map, and $B \times_M B \xrightarrow{\Psi} M$ - a morphism over M. This morphism is called **associative multiplication** if it is associative on the fibers of π, that is, satisfies $\Psi(a, \Psi(b,c)) = \Psi(\Psi(a,b),c)$ for every triple a, b, c in the same fiber.

A section $M \xrightarrow{e} B$ is called **the unit** if the maps $B \xrightarrow{\mathrm{Id}_B \times e} B \times_M B \xrightarrow{\Psi} B$ and $B \xrightarrow{e \times \mathrm{Id}_B} B \times_M B \xrightarrow{\Psi} B$ are equal to Id_B.

A morphism $\nu: B \longrightarrow B$ over M is called **a group inverse** if each of the maps $B \xrightarrow{\Delta} B \times_M B \xrightarrow{\mathrm{Id}_B \times \nu} B \times_M B \xrightarrow{\Psi} B$ and $B \xrightarrow{\Delta} B \times_M B \xrightarrow{\nu \times \mathrm{Id}_B} B \times_M B \xrightarrow{\Psi} B$ is a constant map, mapping b to $e(\pi(b))$.

A map $B \xrightarrow{\pi} M$ equipped with associative multiplication, unit and group inverse is called **a topological group over M**.

Vector spaces over a base

DEFINITION: Let $k=\mathbb{R}$ or $\mathbb{C}$. An abelian topological group $B \xrightarrow{\pi} M$ over M is called **relative vector space over M** if for each continuous k-valued function f there exists a continuous automorphism $\varphi_f: B \longrightarrow B$ of a group B over M which makes each fiber $\pi^{-1}(b)$ into a vector space in such a way that φ_f acts on $\pi^{-1}(b)$ as a multiplication by $\varphi_f(b)$.

REMARK: Let $B \xrightarrow{\pi} M$ be a relative vector space over M, $U \subset M$ an open subset, and $\mathcal{B}(U)$ the space of sections of a map $\pi^{-1}(U) \xrightarrow{\pi} U$. Then **$\mathcal{B}(U)$ defines a sheaf of modules over a sheaf $C^0(M)$ of continuous functions.**

EXAMPLE: Let $S \subset \mathbb{R}^n$ be a subset (not necessarily a smooth submanifold), $s \in S$ a point, and $v \in T_s\mathbb{R}^n$ a vector. We sat that v belongs to a **tangent cone** C_sS if the distance from S to a point $s+tv$ converges to 0 as $t \to 0$ faster than linearly: $\lim\limits_{t\to 0} \frac{d(S,s+tv)}{t}=0$. **Then the set CS of all pairs $(s,v), s\in S, v \in C_sS$ is a relative vector space over S.**

Total space of a vector bundle

DEFINITION: Let $B \longrightarrow M$ be a smooth locally trivial fibration with fiber $\mathbb{R}^n$. Assume that B is equipped with a structure of relative vector space over M, and all the maps used in the definition of a relative vector space are smooth. Then B is called **the total space of a vector bundle.**

REMARK: Let $\pi : B \longrightarrow M$ be a total space of a vector bundle, $U \subset M$ open subset, and $\mathcal{B}(U)$ the space of all smooth sections of $\pi^{-1}(U) \xrightarrow{\pi} U$. **Then $\mathcal{B}$ is a locally free sheaf of $C^\infty M$-modules**.

THEOREM: Every locally free sheaf $C^\infty M$-modules is defined from a total space of a vector bundle, which is determined uniquely by a sheaf.

The proof will be given later.

Fiber of a locally free sheaf

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules on M, $x \in M$ a point, $\mathcal{B}_x$ the space of germs of $\mathcal{B}$ in x, and $\mathfrak{m}_x \subset C^\infty_x M$ the maximal ideal in the ring of germs $C^\infty_x M$ of smooth functions. Define **the fiber** of $\mathcal{B}$ in x as a quotient $\mathcal{B}_x/\mathfrak{m}_x\mathcal{B}_x$. A fiber of $\mathcal{B}$ is denoted $\mathcal{B}|_x$.

REMARK: A fiber of an n-dimensional bundle is an n-dimensional vector space.

REMARK: Let $\mathcal{B} = C^\infty M^n$, and $b \in \mathcal{B}|_x$ a point of a fiber, represented by a germ $\varphi \in \mathcal{B}_x = C^\infty_m M^n$, $\varphi = (f_1, ..., f_n)$. Consider a map Ψ from the set of all fibers $\mathcal{B}$ to $M \times \mathbb{R}^n$, mapping $(x, \varphi = (f_1, ..., f_n))$ to $(f_1(x), ..., f_n(x))$. **Then Ψ is bijective.** Indeed, $\mathcal{B}|_x = \mathbb{R}^n$.

Total space of a vector bundle from its sheaf of sections

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules. Denote the set of all vectors in all fibers of $\mathcal{B}$ over all points of M by $\operatorname{Tot}\mathcal{B}$. Let $U \subset M$ be an open subset of M, with $\mathcal{B}|_U$ a trivial bundle. Using the local bijection $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ we consider topology on $\operatorname{Tot}\mathcal{B}$ induced by open subsets in $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ for all open subsets $U \subset M$ and all trivializations of $\mathcal{B}|_U$. Then $\operatorname{Tot}\mathcal{B}$ is called **a total space of a vector bundle $\mathcal{B}$.**

CLAIM: The space $\operatorname{Tot}\mathcal{B}$ with this topology **is a locally trivial fibration over M, with fiber $\mathbb{R}^n$.** Moreover, it is a relative vector space over M, and **the sheaf of smooth sections of $\operatorname{Tot}\mathcal{B} \longrightarrow M$ is isomorphic to $\mathcal{B}$.**

REMARK: This gives an equivalence between locally free sheaves of $\mathbb{C}^\infty$-modules and the total spaces of vector bundles, defined abstractly in terms of locally trivial fibrations.