Geometry of manifolds

lecture 7

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November 16, 2015

Sheaves (reminder)

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below is a more abstract version of the notion of "sheaf of functions" defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Ringed spaces (reminder)

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A sheaf of functions is a subsheaf in the sheaf of all functions, closed under multiplication.

For simplicity, I assume now that a sheaf of rings is a subsheaf in the sheaf of all functions.

DEFINITION: A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Smooth manifolds (reminder)

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: **Diffeomorphism** of smooth manifolds is a homeomorphism φ which induces an isomorphisms of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Sheaves of modules (reminder)

REMARK: Let $A: \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\varphi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: **A vector bundle** on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Locally trivial fibrations (reminder)

DEFINITION: A smooth map $f: X \longrightarrow Y$ is called **a locally trivial fibration** if each point $y \in Y$ has a neighbourhood $U \ni y$ such that $f^{-1}(U)$ is diffeomorphic to $U \times F$, and the map $f: f^{-1}(U) = U \times F \longrightarrow U$ is a projection. In such situation, F is called **the fiber** of a locally trivial fibration.

DEFINITION: A trivial fibration is a map $X \times Y \longrightarrow Y$.

DEFINITION: A vector bundle on Y is a locally trivial fibration $f: X \longrightarrow Y$ with fiber $\mathbb{R}^n$, with each fiber $V := f^{-1}(y)$ equipped with a structure of a vector space, smoothly depending on $y \in Y$.

THEOREM: This definition is equivalent to the one in terms of sheaves.

Tensor product

DEFINITION: Let V, V' be R-modules, W a free abelian group generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subgroup generated by combinations $rv \otimes v' - v \otimes rv'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$. Define **the tensor product** $V \otimes_R V'$ as a quotient group W/W_1.

EXERCISE: Show that $r \cdot v \otimes v' \mapsto (rv) \otimes v'$ **defines an R-module structure on $V \otimes_R V'$.**

REMARK: Let $\mathcal{F}$ be a sheaf of rings, and $\mathcal{B}_1$ and $\mathcal{B}_2$ be sheaves of locally free $(M, \mathcal{F})$-modules. **Then**

$$U \longrightarrow \mathcal{B}_1(U) \otimes_{\mathcal{F}(U)} \mathcal{B}_2(U)$$

is also a locally free sheaf of modules.

DEFINITION: **Tensor product** of vector bundles is a tensor product of the corresponding sheaves of modules.

EXERCISE: Let B and B' ve vector bundles on M, $B|_x$, $B'|_x$ their fibers, and $B \otimes_{C^\infty M} B'$ their tensor product. **Prove that $B \otimes_{C^\infty M} B'|_x = B|_x \otimes_{\mathbb{R}} B'|_x$.**

Dual bundle and bilinear forms

DEFINITION: Let V be an R-module. **A dual R-module** V^* is $\operatorname{Hom}_R(V,R)$ with the R-module structure defined as follows: $r \cdot h(\dots) \mapsto rh(\dots)$.

CLAIM: Let $\mathcal{B}$ be a vector bundle, that is, a locally free sheaf of $C^\infty M$-modules, and $\operatorname{Tot}\mathcal{B} \xrightarrow{\pi} M$ its total space. Define $\mathcal{B}^*(U)$ as a space of smooth functions on $\pi^{-1}(U)$ linear in the fibers of π. **Then $\mathcal{B}^*(U)$ is a locally free sheaf over $C^\infty(U)$.**

DEFINITION: This sheaf is called **the dual vector bundle**, denoted by B^*. Its fibers are dual to the fibers of B.

DEFINITION: **Bilinear form** on a bundle $\mathcal{B}$ is a section of $(\mathcal{B} \otimes \mathcal{B})^*$. A symmetric bilinear form on a real bundle $\mathcal{B}$ is called **positive definite** if it gives a positive definite form on all fibers of $\mathcal{B}$. Symmetric positive definite form is also called **a metric**. A skew-symmetric bilinear form on $\mathcal{B}$ is called **non-degenerate** if it is non-degenerate on all fibers of $\mathcal{B}$.

Subbundles

DEFINITION: A subbundle $\mathcal{B}_1 \subset \mathcal{B}$ is a subsheaf of modules which is also a vector bundle, and such that the quotient $\mathcal{B}/\mathcal{B}_1$ is also a vector bundle.

DEFINITION: Direct sum $\oplus$ of vector bundles is a direct sum of corresponding sheaves.

EXAMPLE: Let $\mathcal{B}$ be a vector bundle equipped with a metric (that is, a positive definite symmetric form), and $\mathcal{B}_1 \subset \mathcal{B}$ a subbundle. Consider a subset $\operatorname{Tot}\mathcal{B}_1^\perp \subset \operatorname{Tot}\mathcal{B}$, consisting of all $v \in \mathcal{B}|_x$ orthogonal to $\mathcal{B}_1|_x \subset \mathcal{B}|_x$. **Then $\operatorname{Tot}\mathcal{B}_1^\perp$ is a total space of a subbundle, denoted as $\mathcal{B}_1^\perp \subset \mathcal{B}$,** and we have an isomorphism $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_1^\perp$.

REMARK: A total space of a direct sum of vector bundles $\mathcal{B} \oplus \mathcal{B}'$ **is homeomorphic to $\operatorname{Tot}\mathcal{B} \times_M \operatorname{Tot}\mathcal{B}'$.**

EXERCISE: Let $\mathcal{B}$ be a real vector bundle. **Prove that $\mathcal{B}$ admits a metric.**

PROPOSITION: Let $A \subset B$ be a sub-bundle. **Then $B \cong A \oplus C$.**

Proof: Find a positive definite metric on B, and set $C := B^\perp$. ■

Pullback

CLAIM: Let $M_1 \xrightarrow{\varphi} M$ be a smooth map of manifolds, and $B \xrightarrow{\pi} M$ a total space of a vector bundle. **Then $B \times_M M_1$ is a total space of a vector bundle on M_1.**

Proof. Step 1: $B \times_M M_1$ is obviously a relative vector space. Indeed, the fibers of projection $\pi_1: B \times_M M_1 \longrightarrow M_1$ are vector spaces, $\pi_1^{-1}(m_1) = \pi^{-1}(\varphi(m_1))$. It remains only to show that it is locally trivial.

Step 2: Consider an open set $U \subset M$ that $B|_U = U \times \mathbb{R}^n$, and let $U_1 := \varphi^{-1}U$. Then $B \times_U U_1 = U_1 \times \mathbb{R}^n$. **Since M_1 is covered by such U_1, this implies that π_1 is a locally trivial fibration,** and the additive structure smoothly depends on $m_1 \in M_1$. ■

DEFINITION: The bundle $\pi_1: B \times_M M_1 \longrightarrow M_1$ is denoted $\varphi^* B$, and called **inverse image**, or **a pullback** of B.

The Grassmann algebra

DEFINITION: Let V be a vector space. Denote by $\Lambda^i V$ the space of antisymmetric polylinear i-forms on V^*, and let $\Lambda^* V := \bigoplus \Lambda^i V$. Denote by $T^{\otimes i} V$ the algebra of **all** polylinear i-forms on V^* ("tensor algebra"), and let $\operatorname{Alt}: T^{\otimes i} V \longrightarrow \Lambda^i V$ be **the antisymmetrization**,

$$\operatorname{Alt}(\eta)(x_1, ..., x_i) := \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (-1)^{\tilde{\sigma}} \eta(x_{\sigma_1}, ..., x_{\sigma_i})$$

where Σ_i is the group of permutations, and $\tilde{\sigma} = 1$ for odd permutations, and 0 for even. Consider the multiplicative operation ("wedge-product") on $\Lambda^* V$, denoted by $\eta \wedge \nu := \operatorname{Alt}(\eta \otimes \nu)$. The space $\Lambda^* V$ with this operation is called **the Grassmann algebra**.

REMARK: It is an algebra of anti-commutative polynomials.

Prove the properties of Grassmann algebra:

1. $\dim \Lambda^i V := \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.

2. $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$.