Geometry of manifolds

lecture 8

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November 23, 2015

Sheaves of modules (reminder)

REMARK: Let $A: \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\varphi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: **A vector bundle** on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Locally trivial fibrations (reminder)

DEFINITION: A smooth map $f: X \longrightarrow Y$ is called **a locally trivial fibration** if each point $y \in Y$ has a neighbourhood $U \ni y$ such that $f^{-1}(U)$ is diffeomorphic to $U \times F$, and the map $f: f^{-1}(U) = U \times F \longrightarrow U$ is a projection. In such situation, F is called **the fiber** of a locally trivial fibration.

DEFINITION: **A trivial fibration** is a map $X \times Y \longrightarrow Y$.

DEFINITION: **A vector bundle** on Y is a locally trivial fibration $f: X \longrightarrow Y$ with fiber $\mathbb{R}^n$, with each fiber $V := f^{-1}(y)$ equipped with a structure of a vector space, smoothly depending on $y \in Y$.

THEOREM: **This definition is equivalent to the one in terms of sheaves.**

Tensor product (reminder)

DEFINITION: Let V, V' be R-modules, W a free abelian group generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subgroup generated by combinations $rv \otimes v' - v \otimes rv'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$. Define **the tensor product** $V \otimes_R V'$ as a quotient group W/W_1.

EXERCISE: Show that $r \cdot v \otimes v' \mapsto (rv) \otimes v'$ **defines an R-module structure on $V \otimes_R V'$.**

REMARK: Let $\mathcal{F}$ be a sheaf of rings, and $\mathcal{B}_1$ and $\mathcal{B}_2$ be sheaves of locally free $(M, \mathcal{F})$-modules. **Then**

$$U \longrightarrow \mathcal{B}_1(U) \otimes_{\mathcal{F}(U)} \mathcal{B}_2(U)$$

is also a locally free sheaf of modules.

DEFINITION: Tensor product of vector bundles is a tensor product of the corresponding sheaves of modules.

EXERCISE: Let B and B' ve vector bundles on M, $B|_x$, $B'|_x$ their fibers, and $B \otimes_{C^\infty M} B'$ their tensor product. **Prove that $B \otimes_{C^\infty M} B'|_x = B|_x \otimes_{\mathbb{R}} B'|_x$.**

Pullback (reminder)

CLAIM: Let $M_1 \xrightarrow{\varphi} M$ be a smooth map of manifolds, and $B \xrightarrow{\pi} M$ a total space of a vector bundle. **Then $B \times_M M_1$ is a total space of a vector bundle on M_1.**

Proof. Step 1: $B \times_M M_1$ is obviously a relative vector space. Indeed, the fibers of projection $\pi_1 : B \times_M M_1 \longrightarrow M_1$ are vector spaces, $\pi_1^{-1}(m_1) = \pi^{-1}(\varphi(m_1))$. It remains only to show that it is locally trivial.

Step 2: Consider an open set $U \subset M$ that $B|_U = U \times \mathbb{R}^n$, and let $U_1 := \varphi^{-1}U$. Then $B \times_U U_1 = U_1 \times \mathbb{R}^n$. **Since M_1 is covered by such U_1, this implies that π_1 is a locally trivial fibration,** and the additive structure smoothly depends on $m_1 \in M_1$. ■

DEFINITION: The bundle $\pi_1 : B \times_M M_1 \longrightarrow M_1$ is denoted $\varphi^* B$, and called **inverse image**, or **a pullback** of B.

Algebras represented by generators and relations

DEFINITION: Let V be a vector space. **Free** or **tensor algebra** generated by V is an algebra $T(V) := \bigoplus_i V^{\otimes i}$ with multiplivation given by $x \cdot y = x \otimes y$. The zero component $V^{\otimes 0}$ is identified with the ground field. Thus, $T(V)$ is an algebra with unit.

DEFINITION: Let V be a vector space over the ground field k, called **"the space of generators"**, and $W \subset T(V)$ another space called **"the space of relations"**. Consider the quotient space $A := \frac{T(V)}{T(V)WT(V)}$, where $T(V)WT(V)$ is a subspace of $T(V)$ generated by vectors vwv', where $w \in W$, $v, v' \in T(V)$. Then A is is equipped with a natural structure of an algebra with unit, in such a way that the quotient map $T(V) \longrightarrow A$ is an isomorphism. The algebra A is called **algebra generated by generators and relations**.

EXAMPLE: Let V be a vector space with a bilinear symmetric form $g : V \otimes V \longrightarrow \mathbb{R}$. Consider the algebra $\mathrm{Cl}(V)$ generated by V, with relations

$$v_1 \cdot v_2 + v_2 \cdot v_1 = g(v_1, v_2) \cdot 1,$$

for all $v_1, v_2 \in V$. This algebra is called **Clifford algebra** over k.

Graded algebras

DEFINITION: An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always assumed to belong in A^0.

EXAMPLE: The tensor algebra $T(V)$ and the polynomial algebra $\operatorname{Sym}^*(V)$ are obviously graded.

DEFINITION: A subspace $W \subset A^*$ of a graded algebra is called **graded** if W is a direct sum of components $W^i \subset A^i$.

EXERCISE: Let $W \subset T(V)$ be a graded subspace. Then **the algebra generated by V with relation space W is also graded.**

The Grassmann algebra

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a graded subspace, generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

REMARK: Grassmann algebra is a a Clifford algebra with the symmetric form $g = 0$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

Signature of a permutation

CLAIM: Let $x_1, x_2, ...$ be a basis in $V \cong \Lambda^1 V$. Then **the set of vectors $x_{i_1} \wedge x_{i_2} \wedge x_{i_3} \wedge \cdots$, for all $i_1 < i_2 < i_3 < ...$ is a basis in $\Lambda^*(V)$.**

COROLLARY: Let V be a d-dimensional vector space. **Then $\dim \Lambda^d(V) = \binom{n}{d}$.** In particular, $\dim \Lambda^d V = 1$.

DEFINITION: The space $\Lambda^d V$ is called **the space of determinant vectors on V**.

CLAIM: Let V be a d-dimensional vector space, $x_1, x_2, ..., x_d$ its basis, and $\det := x_1 \wedge x_2 \wedge x_3 \cdots \wedge x_d$ the corresponding determinant vector in $\Lambda^d V$. For a given permutation $I = (i_1, i_2, ..., i_d)$ consider a vector $I(\det) := x_{i_1} \wedge x_{i_2} \wedge x_{i_3} \cdots \wedge x_{i_d}$. **Then $I(\det) = \pm \det$.** This correspondence gives a homomorphism σ from the group S_d of permutations to $\{\pm 1\}$.

REMARK: This homomorphism maps a product of odd number of transpositions to -1 and a product of even number of transpositions to 1.

DEFINITION: The number $\sigma(I)$ is called **signature** of a permitation I. Permutation I is called **odd** if $\sigma(I) = -1$ and **even** if $\sigma(I) = 1$.

Antisymmetrization

DEFINITION: Let V be a vector space, $T^{\otimes i}V$ the i-th tensor power of V, and $\operatorname{Alt} : T^{\otimes i}V \longrightarrow \Lambda^i V$ be **the antisymmetrization**,

$$\operatorname{Alt}(\eta)(x_1, ..., x_i) := \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (-1)^{\tilde\sigma} \eta(x_{\sigma_1}, ..., x_{\sigma_i})$$

where Σ_i is the group of permutations, and $\tilde\sigma = 1$ for odd permutations, and 0 for even. We say that a vector $\eta \in V^{\otimes d}$ is **totally antisymmetric** if $\eta = \operatorname{Alt}(\eta)$.

EXERCISE: Let $\eta \in V^{\otimes d}$ be a vector which satisfies $\eta = \frac{1}{d!}(-1)^{\tilde\sigma} \sum_{I \in S_d} I(\eta)$. **Prove that $I(\eta) = (-1)^{\tilde\sigma}\eta$ for any permutation $\sigma \in S_d$.**

REMARK: This implies that $\operatorname{Alt}(\operatorname{Alt}(\eta)) = \operatorname{Alt}(\eta)$ **for any $\eta \in V^{\otimes d}$.**

CLAIM: Let $W \subset V \otimes V$ be the space of relations of Grassmann algebra defined above. **Then $\operatorname{Alt}(T(V) \cdot W \cdot T(V)) = 0$.**

COROLLARY: This defines **a natural linear map Ψ from $\Lambda^*(V)$ to the space $\operatorname{im}\operatorname{Alt}$ of totally antisymmetric tensors.**

Antisymmetric tensors and Grassmann algebra

CLAIM: The map Ψ gives an isomorphism of $\Lambda^*(V)$ and $\operatorname{im}\mathsf{Alt}$.

Proof. Step 1: Since $\operatorname{im}\mathsf{Alt}$ is generated by antisymmetrizations of monomials $x_{i_1}\otimes x_{i_2}\otimes ...\otimes x_{i_d}$, and all these monomials belong to $\operatorname{im}\Psi$, Ψ is surjective.

Step 2: These antisymmetrizations are linearly independent in the tensor algebra $T(V)$, because they all contain different tensor monomials. This implies that the basis in $\Lambda^*(V)$ is mapped into linearly independent elements of $\operatorname{im}\mathsf{Alt}$, and Ψ is injective. ■

REMARK: From now on, **we identify $\Lambda^*(V)$ and the space of totally antisymmetric tensors.**

REMARK: This identification **defines multiplicative structure on the space $\operatorname{im}\mathsf{Alt}$ of totally antisymmetric tensors.**

CLAIM: The multiplicative structure on $\operatorname{im}\mathsf{Alt}$ can be written as follows. Given totally antisymmetric tensors $\alpha,\beta\in\operatorname{im}\mathsf{Alt}$, **to find $\alpha\wedge\beta\in\operatorname{im}\mathsf{Alt}=\Lambda^*(V)$, we muptiply α and β in $T(V)$ and apply Alt.**

Proof: It suffices to check this on monomials. ■

Determinant

REMARK: Let W be a one-dimensional vector space over k. **Then** End W **is naturally isomorphic to k.**

REMARK: Let $A \in \operatorname{End}(V)$ be a linear endomorphism of a vector space V. Then **the action of A on $V \cong \Lambda^1 V$ is uniquely extended to a multiplicative homomorphism of the algebra $\Lambda^* V$.**

DEFINITION: Let V be a d-dimensional vector space and $A \in \operatorname{End}(V)$. Consider the induced endomorphism of the space of determinant vectors $\Lambda^d(V)$ denoted as $\det A \in \operatorname{End}(\Lambda^d(V))$. Since $\Lambda^d(V)$ is 1-dimensional, the space $\operatorname{End}(\Lambda^d(V))$ is naturally identified with k. This allows to consider $\det A$ as a number, that is, an element of k. This number is called **determinant** of A.

REMARK: From the definition it is clear that det **defines a homomorphism from the group $GL(V)$ if invertible matrices to the multiplicative group k^* of the ground field.**

Determinant bundle

DEFINITION: A line bundle is a 1-dimensional vector bundle.

EXERCISE: Let M be a simply connected manifold. **Prove that any real line bundle on M is trivial.**

DEFINITION: Let B be a vector bundle of rank n, and $\Lambda^n B$ its top exterior product. This bundle is called **determinant bundle** of B.

REMARK: It is a line bundle.

REMARK: Let M be an n-manifold, and $\Lambda^n TM$ a determinant bundle of its tangent bundle. Prove that $\Lambda^n TM$ **is trivial if and only if M is orientable.**

DEFINITION: A real vector bundle is called **orientable** if its determinant bundle is trivial.

DEFINITION: A manifold is called **orientable** if its tangent bundle is orientable.

Cotangent bundle

DEFINITION: Let M be a smooth manifold, TM the tangent bundle, and $\Lambda^1 M = T^*M$ its dual bundle. It is called **cotangent bundle** of M. Sections of T^*M are called **1-forms** or **covectors** on M. For any $f \in C^\infty M$, consider a functional $TM \longrightarrow C^\infty M$ obtained by mapping $X \in TM$ to a derivation of f: $X \longrightarrow D_X(f)$. Since this map is linear in X, it defines a section $df \in T^*M$ called **the differential** of f.

CLAIM: **$\Lambda^1 M$ is generated as a $C^\infty M$-module by $d(C^\infty M)$.**

Proof: Locally in coordinates $x_1, ..., x_n$ this is clear, because the covectors $dx_1, ..., dx_n$ dive a basis in T^*M dual to the basis $\frac{d}{dx_1}, ..., \frac{d}{dx_n}$ in TM. ■

DEFINITION: Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^*M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

REMARK: $\Lambda^0 M = C^\infty M$.

De Rham algebra

DEFINITION: Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on V. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

DEFINITION: Let $\bigotimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge: \ \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^*M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x \in M$ **is identified with the Grassmann algebra $\Lambda^* T_x^* M$.** This identification is compatible with the Grassmann product.

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha \in \Lambda^* \mathbb{R}^n$ a monomial obtained as a product of several dt_i: $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$ $i_1 < i_2 < ... < i_k$. Then α is called **a coordinate monomial**.