Geometry of manifolds

lecture 9

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Sheaves of modules (reminder)

REMARK: Let $A: \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\varphi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: **A vector bundle** on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Tensor product (reminder)

DEFINITION: Let V, V' be R-modules, W a free abelian group generated by $v\otimes v'$, with $v\in V, v'\in V'$, and $W_1\subset W$ a subgroup generated by combinations $rv\otimes v' - v\otimes rv'$, $(v_1+v_2)\otimes v' - v_1\otimes v' - v_2\otimes v'$ and $v\otimes(v_1'+v_2') - v\otimes v_1' - v\otimes v_2'$. Define **the tensor product** $V\otimes_R V'$ as a quotient group W/W_1.

EXERCISE: Show that $r\cdot v\otimes v'\mapsto (rv)\otimes v'$ **defines an R-module structure on $V\otimes_R V'$.**

REMARK: Let $\mathcal{F}$ be a sheaf of rings, and $\mathcal{B}_1$ and $\mathcal{B}_2$ be sheaves of locally free $(M,\mathcal{F})$-modules. **Then**

$$U\longrightarrow \mathcal{B}_1(U)\otimes_{\mathcal{F}(U)}\mathcal{B}_2(U)$$

is also a locally free sheaf of modules.

DEFINITION: **Tensor product** of vector bundles is a tensor product of the corresponding sheaves of modules.

EXERCISE: Let B and B' ve vector bundles on M, $B|_x$, $B'|_x$ their fibers, and $B\otimes_{C^\infty M} B'$ their tensor product. **Prove that $B\otimes_{C^\infty M} B'|_x = B|_x\otimes_{\mathbb{R}} B'|_x$.**

The Grassmann algebra (reminder)

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a graded subspace, generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

Antisymmetrization (reminder)

DEFINITION: Let V be a vector space, $T^{\otimes i}V$ the i-th tensor power of V, and $\mathsf{Alt}: T^{\otimes i}V \longrightarrow \Lambda^i V$ be **the antisymmetrization**,

$$\mathsf{Alt}(\eta)(x_1, ..., x_i) := \frac{1}{i!}\sum_{\sigma\in\Sigma_i}(-1)^{\tilde\sigma}\eta(x_{\sigma_1}, ..., x_{\sigma_i})$$

where Σ_i is the group of permutations, and $\tilde\sigma = 1$ for odd permutations, and 0 for even. We say that a vector $\eta \in V^{\otimes d}$ is **totally antisymmetric** if $\eta = \mathsf{Alt}(\eta)$.

EXERCISE: Let $\eta \in V^{\otimes d}$ be a vector which satisfies $\eta = \frac{1}{d!}(-1)^{\tilde\sigma}\sum_{I\in S_d} I(\eta)$. **Prove that $I(\eta) = (-1)^{\tilde\sigma}\eta$ for any permutation $\sigma \in S_d$.**

REMARK: This implies that $\mathsf{Alt}(\mathsf{Alt}(\eta)) = \mathsf{Alt}(\eta)$ **for any $\eta \in V^{\otimes d}$.**

CLAIM: Let $W \subset V \otimes V$ be the space of relations of Grassmann algebra defined above. **Then $\mathsf{Alt}(T(V)\cdot W \cdot T(V)) = 0$.**

COROLLARY: This defines **a natural linear map Ψ from $\Lambda^*(V)$ to the space $\operatorname{im}\mathsf{Alt}$ of totally antisymmetric tensors.**

Antisymmetric tensors and Grassmann algebra (reminder)

CLAIM: The map Ψ gives an isomorphism of $\Lambda^*(V)$ and $\operatorname{im} \mathsf{Alt}$**.**

REMARK: From now on, **we identify $\Lambda^*(V)$ and the space of totally antisymmetric tensors.**

REMARK: This identification **defines multiplicative structure on the space** $\operatorname{im} \mathsf{Alt}$ **of totally antisymmetric tensors.**

CLAIM: The multiplicative structure on $\operatorname{im} \mathsf{Alt}$ can be written as follows. Given totally antisymmetric tensors $\alpha, \beta \in \operatorname{im} \mathsf{Alt}$, **to find** $\alpha \wedge \beta \in \operatorname{im} \mathsf{Alt} = \Lambda^*(V)$**, we muptiply α and β in $T(V)$ and apply** Alt**.**

Proof: It suffices to check this on monomials. ■

Cotangent bundle

DEFINITION: Let M be a smooth manifold, TM the tangent bundle, and $\Lambda^1 M = T^*M$ its dual bundle. It is called **cotangent bundle** of M. Sections of T^*M are called **1-forms** or **covectors** on M. For any $f \in C^\infty M$, consider a functional $TM \longrightarrow C^\infty M$ obtained by mapping $X \in TM$ to a derivation of f: $X \longrightarrow D_X(f)$. Since this map is linear in X, it defines a section $df \in T^*M$ called **the differential** of f.

CLAIM: $\Lambda^1 M$ is generated as a $C^\infty M$-module by $d(C^\infty M)$.

Proof: Locally in coordinates $x_1, ..., x_n$ this is clear, because the covectors $dx_1, ..., dx_n$ dive a basis in T^*M dual to the basis $\frac{d}{dx_1}, ..., \frac{d}{dx_n}$ in TM. ■

DEFINITION: Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^*M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

REMARK: $\Lambda^0 M = C^\infty M$.

De Rham algebra

DEFINITION: Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on V. Define the **tensor multiplication** $\alpha\otimes\beta$ as

$$\alpha\otimes\beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

DEFINITION: Let $\bigotimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!}\sum_{\sigma\in \mathsf{Sym}_n}(-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge: \Lambda^i M\times\Lambda^j M\longrightarrow \Lambda^{i+j}M$ as $\alpha\wedge\beta := \Pi(\alpha\otimes\beta)$, where $\alpha\otimes\beta$ is a section $\Lambda^i M\otimes\Lambda^j M\subset\bigotimes_{i+j}T^*M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x\in M$ **is identified with the Grassmann algebra $\Lambda^* T^*_x M$.** This identification is compatible with the Grassmann product.

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha\in\Lambda^*\mathbb{R}^n$ a monomial obtained as a product of several dt_i: $\alpha = dt_{i_1}\wedge dt_{i_2}\wedge ...\wedge dt_{i_k}$ $i_1 < i_2 < ... < i_k$. Then α is called **a coordinate monomial**.

De Rham differential

DEFINITION: De Rham differential $d: \Lambda^* M \longrightarrow \Lambda^{*+1} M$ is an $\mathbb{R}$-linear map satisfying the following conditions.

* For each $f \in \Lambda^0 M = C^\infty M$, $d(f) \in \Lambda^1 M$ is equal to the differential $df \in \Lambda^1 M$.
* **(Leibnitz rule)** $d(a \wedge b) = da \wedge b + (-1)^j a \wedge db$ for any $a \in \Lambda^i M, b \in \Lambda^j M$.
* $d^2 = 0$.

REMARK: A map on a graded algebra which satisfies the Leibnitz rule above is called **an odd derivation**.

REMARK: The following two lemmas are needed to prove uniqueness of de Rham differential.

LEMMA: Let $A = \bigoplus A^i$ be a graded algebra, $B \subset A$ a set of multiplicative generators, and $D_1, D_2: A \longrightarrow A$ two odd derivations which are equal on B. **Then $D_1 = D_2$.** ■

LEMMA: $\Lambda^* M$ is generated by $C^\infty M$ and $d(C^\infty M)$.

Proof: By definition, $\Lambda^* M$ is generated by $\Lambda^0 M = C^\infty M$ and $\Lambda^1 M$. However, $d(C^\infty M)$ generate $\Lambda^1 M$, as shown above. ■

De Rham differential: uniqueness and existence

THEOREM:
De Rham differential is uniquely determined by these axioms.

Proof: De Rham differential is an odd derivation. Its value on $C^\infty M$ is defined by the first axiom. On $d(C^\infty M)$ de Rham differential valishes, because $d^2 = 0$. ■

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, α_i coordinate monomials, and $\alpha := \sum f_i \alpha_i$. Define $d(\alpha) := \sum_i \sum_j \frac{df_i}{dt_j} dt_j \wedge \alpha_i$.

EXERCISE:
Check that d satisfies the properties of de Rham differential.

COROLLARY: **De Rham differential exists on any smooth manifold.**

Proof: Locally, de Rham differential d exists, as follows from the construction above. Since d is unique, it is compatible with restrictions. **This means that d defines a sheaf morphism.** Restricting this sheaf morphism to global sections, we obtain de Rham differential on $\Lambda^* M$. ■

Superalgebras

DEFINITION: Let $A^* = \oplus_{i\in\mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij} ba$ for all $a \in A^i, b \in A^j$.

EXAMPLE: Grassmann algebra $\Lambda^* V$ is clearly supercommutative.

DEFINITION: Let A^* be a graded commutative algebra, and $D: A^* \longrightarrow A^{*+i}$ be a map which shifts grading by i. It is called a **graded derivation** if $D(ab) = D(a)b + (-1)^{ij} aD(b)$, for each $a \in A^j$.

REMARK: If i is even, graded derivation is a usual derivation. If it is even, it an odd derivation.

DEFINITION: Let M be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **convolution with a vector field** $i_X: \Lambda^i M \longrightarrow \Lambda^{i-1} M$, mapping an i-form α to an $(i-1)$-form $v_1, ..., v_{i-1} \longrightarrow \alpha(X, v_1, ..., v_{i-1})$

EXERCISE: Prove that i_X is an odd derivation.

Supercommutator

DEFINITION: Let A^* be a graded vector space, and $E: \ A^* \longrightarrow A^{*+i}$, $F: \ A^* \longrightarrow A^{*+j}$ operators shifting the grading by i, j. Define **the supercommutator** $\{E, F\} := EF - (-1)^{ij} FE$.

DEFINITION: An endomorphism which shifts a grading by i is called **even** if i is even, and **odd** otherwise.

EXERCISE: Prove that a supercommutator satisfies **graded Jacobi identity,**

$$\{E,\{F,G\}\} = \{\{E,F\},G\} + (-1)^{\tilde E \tilde F}\{F,\{E,G\}\}$$

where $\tilde E$ and $\tilde F$ are 0 if E, F are even, and 1 otherwise.

REMARK: There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. **Each time when in commutative case two letters E, F are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde E \tilde F}$.**

EXERCISE: **Prove that a supercommutator of superderivations is again a superderivation.**

Pullback of a differential form

DEFINITION: Let $M \xrightarrow{\varphi} N$ be a morphism of smooth manifolds, and $\alpha \in \Lambda^i N$ be a differential form. Consider an i-form $\varphi^*\alpha$ taking value

$$\alpha\Big|_{\varphi(m)}(D_\varphi(x_1), ...D_\varphi(x_i))$$

on $x_1, ..., x_i \in T_m M$. It is called **the pullback of α**. If $M \xrightarrow{\varphi} N$ is a closed embedding, the form $\varphi^*\alpha$ is called **the restriction** of α to $M \hookrightarrow N$.

LEMMA: (*) Let $\Psi_1, \Psi_2 : \Lambda^* N \longrightarrow \Lambda^* M$ be two maps which satisfy graded Leibnitz identity, supercommutes with de Rham differential, and satisfy $\Psi_1|_{C^\infty M} = \Psi_2|_{C^\infty M}$. **Then $\Psi_1 = \Psi_2$.**

Proof: The algebra $\Lambda^* M$ is generated multiplicatively by $C^\infty M$ and $d(C^\infty M)$; restrictions of Ψ_i to these two spaces are equal. ■

CLAIM: Pullback commutes with the de Rham differential.

Proof: Let $d_1, d_2 : \Lambda^* N \longrightarrow \Lambda^{*+1} M$ be the maps $d_1 = \varphi^* \circ d$ and $d_2 = d \circ \varphi^*$. **These maps satisfy the Leibnitz identity, and they are equal on $C^\infty M$.** The super-commutator $\delta := \{d_i, d\}$ is equal to $d \circ \varphi^* \circ d$, it commutes with d, and equal 0 on functions. By Lemma (*), $\delta = 0$. Then d_i supercommutes with d. Applying Lemma (*) again, we obtain that $d_1 = d_2$. ■