Differential geometry: test assignment 1

Rules: Please solve this in class, 16:00-18:00, Monday 28.09.2015, and bring the results to my pigeonhole in the mail room at 9-th floor.

Definition 1.1. A topological space is called **connected** if it cannot be represented as a union of non-empty, non-intersecting open subsets.

Exercise 1.1. Prove that any infinite, countable metric space is not connected.

Exercise 1.2. Let $M := \mathbb{R}^2 \backslash \mathbb{Q}^2$. Prove that M is connected.

Exercise 1.3. Let $Z \subset \mathbb{R}^n$ be a countable set. Construct a function $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ which is continuous at $x \notin Z$ and discontinuous at Z.

Exercise 1.4. Let $f_i : [0,1] \longrightarrow [0,1]$ be a sequence of continuous functions, and $f(z) := \lim_i f_i(z)$. Prove that f is continuous, or find a counterexample.

Exercise 1.5. A function f on a metric space is called **1-Lipschitz** if

$$|f(x) - f(y)| \leqslant d(x,y).$$

Prove that any metric space admits a non-constant 1-Lipschitz function.