Geometry 3: Vector fields and derivations

Rules: Exam problems would be similar to ones marked with ! sign. It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

3.1 Derivations of a ring

Remark 3.1. All rings in these handouts are assumed to be commutative and with unit. Algebras are associative, but not necessarily commutative (such as the matrix algebra). **Rings over a field** k are rings containing a field k.

Definition 3.1. Let R be a ring over a field k. A k-linear map $D: R \longrightarrow R$ is called **a derivation** if it satisfies **the Leibnitz equation** $D(fg) = D(f)g + gD(f)$. The space of derivations is denoted as $\operatorname{Der}_k(R)$.

Exercise 3.1. Let $D \in \operatorname{Der}_k(R)$. Prove that $D\big|_k = 0$.

Exercise 3.2. Let D_1, D_2 be derivations. Prove that the commutator $[D_1, D_2] := D_1D_2 - D_2D_1$ is also a derivation.

Exercise 3.3 (!). Let $K \supset k$ be a field which contains a field k of characteristic 0, and is finite-dimensional over k (such fields K are called **finite extensions** of k). Find the space $\operatorname{Der}_k(K)$.

Exercise 3.4 (*). Is it true if $\operatorname{char} k = p$?

Exercise 3.5. Consider a ring $k[\varepsilon]$, given by a relation $\varepsilon^2 = 0$. Find $\operatorname{Der}_k(k[\varepsilon])$.

Exercise 3.6 (*). Find all rings R over $\mathbb{C}$ such that R is finite-dimensional over $\mathbb{C}$, and $\operatorname{Der}_{\mathbb{C}}(R) = 0$.

Exercise 3.7 (**). Let $D \in \operatorname{Der}_k(K)$ be a derivation of a field K over k, $\operatorname{char} k = 0$, and $[K' : K]$ a finite field extension. Prove that D can be extended to a derivation $D' \in \operatorname{Der}_k(K')$.

Exercise 3.8. Let $D \in \operatorname{Der}_k(R)$ be a derivation, and $I \subset R$ an ideal. Prove that $D(I^k) \subset I^{k-1}$.

3.2 Modules over a ring

Definition 3.2. Let R be a ring over a field k. **An R-module** is a vector space V over k, equipped with an algebra homomorphism $R \longrightarrow \operatorname{End}(V)$, where $\operatorname{End}(V)$ denotes the endomorphism algebra of V, that is, the matrix algebra.

Exercise 3.9. Let R be a field. Prove that R-modules are the same as vector spaces over R.

Remark 3.2. An R-module is a group, equipped with an operation of "multiplication by elements of R", and satisfying the same axioms of distributivity and associativity as in the definition of a vector space.

Remark 3.3. Homomorphisms, isomorphisms, submodules, quotient modules, direct sums of modules are defined in the same way as for the vector spaces. A ring R is itself an R-module. A direct sum of n copies of R is denoted R^n. Such R-module is called **a free R-module**.

Remark 3.4. R-submodules in R are the same as ideals in R.

Definition 3.3. A ring R is called **a principal ideal ring** if all non-zero submodules of R are isomorphic to R.

Exercise 3.10. Prove that R is a principal ideal ring iff R has no zero divisors, and all ideals in R are **principal**, that is, are of form Rx, for some non-invertible $x \in R$.

Exercise 3.11. Are these rings principal ideal rings?

a. $R = \mathbb{C}[t]$

b. (!) $R = \mathbb{C}[t_1, t_2]$

c. (*) $R := \mathbb{R}[x, y]/(x^2 + y^2 = -1)$.

Definition 3.4. Finitely generated R-module is a quotient module of R^n.

Exercise 3.12. Find a finitely generated, non-free R-module for $R = \mathbb{C}[t]$.

Definition 3.5. A Noetherian ring is a ring R with all ideals finitely generated as R-modules.

Exercise 3.13 (*). Let R be a Noetherian ring. Prove that any submodule of a finitely generated R-module is finitely generated.

Exercise 3.14. Consider a ring R of germs of smooth functions in a point $x \in \mathbb{R}^n$, and let K be an ideal of all functions with all derivatives of all orders vanishing. Show that this ideal is not principal.

Exercise 3.15 (*). Prove that $R = \mathbb{C}^\infty\mathbb{R}^n$ is not finitely generated.

3.3 Vector fields

Remark 3.5. Let R be a ring over k. The space $\operatorname{Der}_k(R)$ of derivations is also an R-module, with multiplicative action of R given by $rD(f) = rD(f)$.

Exercise 3.16. Let $R = k[t_1, .., t_k]$ be a polynomial ring. Prove that $\operatorname{Der}_k(R)$ is a free R-module isomorphic to R^n, with generators $\frac{d}{dt_1}, \frac{d}{dt_2}, ..., \frac{d}{dt_n}$.

Hint. Construct a map $\operatorname{Der}_k(R) \longrightarrow R^n$,

$$D \longrightarrow (D(t_1), D(t_2), ..., D(t_n))$$

and prove that it is an isomorphism of R-modules.

Exercise 3.17 (*). Let $R = k(t_1, .., t_k)$ be a ring of rational functions, that is, the ring of functions $\frac{P}{Q}$, where P and $Q \in k(t_1, .., t_k)$ are arbitrary polynomials, $Q \neq 0$. Prove that $\operatorname{Der}_k(R)$ is a free R-module, isomorphic to R^n.

Exercise 3.18 (!). Prove the **Hadamard's lemma**: Let f be a smooth function f on $\mathbb{R}^n$, and x_i the coordinate functions. Then $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, for some smooth $g_i \in C^\infty\mathbb{R}^n$.

Hint. Consider a function $h(t) \in C^\infty\mathbb{R}^n$, $h(t) = f(tx)$. Then $\frac{dh}{dt} = \sum \frac{df(tx)}{dx_i}(tx)x_i$. Integrating this expression over t, obtain $f(x) - f(0) = \sum_i x_i \int_0^1 \frac{df(tx)}{dx_i}(tx)dt$.

Definition 3.6. Consider coordinates $t_1, ..., t_n$ on $\mathbb{R}^n$, and let

$$\operatorname{Der}(C^\infty\mathbb{R}^n) \xrightarrow{\Pi} (C^\infty\mathbb{R}^n)^n,$$

map D to $(D(t_1), D(t_2), ..., D(t_n))$.

Exercise 3.19. Prove that Π is surjective.

Exercise 3.20. Prove that $\Pi(D) = 0 \Leftrightarrow D(P) = 0$ for each $P(t_1, ..., t_n)$.

Exercise 3.21. Let $\mathfrak{m}_x \subset C^\infty\mathbb{R}^n$ be an ideal of all smooth functions vanishing at $x \in \mathbb{R}^n$. Prove that it is maximal.

Exercise 3.22. Let f be a smooth function on $\mathbb{R}^n$ satisfying $f(x) = 0$ and $f'(x) = 0$. Prove that $f \in \mathfrak{m}_x^2$.

Hint. Use the Hadamard's Lemma.

Exercise 3.23 (!). Let $D \in \operatorname{Der}_{\mathbb{R}}(C^\infty\mathbb{R}^n)$ be a derivation, satisfying $D \in \ker\Pi$ (that is, vanishing on coordinate functions). Prove that for all $f \in C^\infty\mathbb{R}^n$, and all $x \in \mathbb{R}^n$, one has $D(f) \in \mathfrak{m}_x$.

Hint. Use the previous exercise and Exercise 3.8.

Exercise 3.24 (!). Prove that the map

$$\operatorname{Der}(C^\infty\mathbb{R}^n) \xrightarrow{\ \Pi\ } (C^\infty\mathbb{R}^n)^n$$

is an isomorphism

Hint. Use the previous exercise.

Exercise 3.25 (**). Find a non-trivial element $\gamma \in \operatorname{Der}_{\mathbb{R}}(C^0\mathbb{R})$ in the space of derivations of continuous functions, or prove that it is empty.

Exercise 3.26 (**). Find a non-trivial element $\gamma \in \operatorname{Der}_{\mathbb{R}}(C^1\mathbb{R})$ in the space of derivations of the ring of differentiable functions of class C^1, or prove that it is empty.