Geometry 4: Germs and sheaves

Rules: Exam problems would be similar to ones marked with ! sign. It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

4.1 Direct limit

Definition 4.1. Commutative diagram of vector spaces is given by the following data. First, there is a directed graph (graph with arrows). For each vertex of this graph (also called a diagram) one gives a vector space, and each arrow corresponds to a homomorphism of the associated vector spaces. These homomorphism are compatible, in the following way. Whenever there exist two ways of going from one vertex to another, the compositions of the corresponding arrows are equal.

Remark 4.1. A **neighbourhood** of a subset $X \subset M$ is an open subset containing X.

Exercise 4.1. Let $(M, \mathcal{F})$ be a space ringed by a sheaf of functions, $x \in M$ a point, $\{U_i\}$ the set of all neighbourhoods of x. Consider a diagram with the set of vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. Prove that the space of sections $\mathcal{F}(U_i)$ with homomorphisms given by restrictions form a commutative diagram.

Definition 4.2. Let $\mathcal{C}$ be a commutative diagram of vector spaces A, B – vector spaces corresponding to two vertices of a diagram, and $a \in A, b \in B$ elements of these vector spaces. Write $a \sim b$ if a and b are mapped to the same element $d \in D$ by a composition of arrows from $\mathcal{C}$. Let $\sim$ be an equivalence relation generated by such $a \sim b$.

Exercise 4.2. a. Let $A \xrightarrow{\phi} B$ be a diagram of two spaces and one arrow. Prove that $b \sim b'$ is equivalent to $b = b'$ for each $b, b' \in B$.

b. Let $A \xrightarrow{\phi} B$, $A \longrightarrow 0$ be a diagram of three spaces, with ϕ injective. Prove that for each $b, b' \in B$, $b \sim b'$ is equivalent to $b - b' \in \operatorname{im}\phi$.

Definition 4.3. Let $\{C_i\}$ be a set of vector spaces associated with the vertices of a commutative diagram $\mathcal{C}$, and $E \subset \bigoplus_i C_i$ a subspace generated by the vectors $(x - y)$, where $x \sim y$. A quotient $\bigoplus_i C_i/E$ is called **a direct limit** of a diagram $\{C_i\}$. The same notion is also called **colimit** and **inductive limit**. Direct limit is denoted $\lim\limits_{\to}$.

Exercise 4.3. Let $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ be a diagram with all arrows injective. Prove that $\lim\limits_{\to} C_i$ is a union of all C_i.

Exercise 4.4. Let $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ... \longrightarrow C_n$ be a diagram. Prove that $\lim\limits_{\to} C_i = C_n$.

Exercise 4.5. Find an example of a diagram $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ where all spaces C_i are non-zero, and the colimit $\lim\limits_{\to} C_i$ vanishes.

Exercise 4.6 (*). Find an example of a diagram $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ where all spaces C_i are non-zero, all arrows are also non-zero, and the colimit $\lim\limits_{\to} C_i$ vanishes.

Definition 4.4. A diagram $\mathcal{C}$ is called **filtered** if for any two vertices C_i, C_j, there exists a third vertex C_k, and sequences of arrows leading from C_i to C_k and from C_j to C_k.

Exercise 4.7. Let $\mathcal{C}$ be a commutative diagram of vector spaces C_i, with all C_i equipped with a ring structure, and all arrows ring homomorphisms. Suppose that the diagram $\mathcal{C}$ is filtered. Prove that $\lim\limits_{\to} C_i$ is a ring, equipped with natural ring homomorphisms $C_i \longrightarrow \lim\limits_{\to} C_i$.

4.2 A ring of germs of a sheaf of functions

Definition 4.5. Let $M, \mathcal{F}$ be a ringed space, $x \in M$ its point, and $\{U_i\}$ the set of all its neighbourhoods. Consider a commutative diagram with vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. For each vertex U_i we take a vector space of sections $\mathcal{F}(U_i)$, and for each arrow the corresponding restriction map. The direct limit of this diagram is called **the ring of germs of the sheaf $\mathcal{F}$ in** x.

Remark 4.2. This limit is indeed a ring, as follows from the previous exercise.

Remark 4.3. As a special case of this definition, we obtain rings of germs of smooth functions, real analytic functions, continuous, C^i and so on.

Exercise 4.8. Let $\mathcal{F}$ be a sheaf of functions on a manifold such that all its germs are zero. Prove that $\mathcal{F}$ is a zero sheaf.

Definition 4.6. A constant sheaf $\mathbb{R}_M$ is a sheaf of functions which are constant on each connected $U \subset M$.

Exercise 4.9. Prove that a ring of germs of a constant sheaf at each point is $\mathbb{R}$.

Exercise 4.10 (*). Let $\mathcal{F}$ be a sheaf of $\mathbb{R}$-valued functions on M, such that all its germs are isomorphic to $\mathbb{R}$. Prove that it is constant.

Definition 4.7. An ideal in a ring R is an abelian subgroup $I \subsetneq R$, such that for all $x \in R, a \in I$, the product xa belongs to I.

Remark 4.4. A quotient space R/I is a ring (prove this). Also, for any ring homomorphism, its kernel is an ideal.

Definition 4.8. A maximal ideal is an ideal $I \subset R$, such that for any other ideal $I' \supsetneq I$, $I' \ni 1$.

Exercise 4.11. Show that any ideal is contained in a maximal ideal (use Zorn's lemma).[1]

Exercise 4.12. Show that an ideal $I \subset R$ is maximal if and only if the quotient R/I is a field.

Exercise 4.13 (*). Find all maximal ideals in the ring of smooth functions on a compact manifold.

Definition 4.9. A ring is called **local** if it contains only one maximal ideal.

Exercise 4.14. Prove that a ring of rational numbers $\frac{m}{n}$, where m, n are integer, and n odd, is local. Find its quotient by the maximal ideal.

Exercise 4.15. Let F be a ring of rational functions (functions $\frac{P}{Q}$, where P and $Q \in \mathbb{C}[t_1, ..., t_n]$ are polynomials) without a pole in 0. Show that this ring is local. Find its quotient by a maximal ideal.

Exercise 4.16 (!). Are the following rings local?

a. The ring of germs of smooth functions.

b. The ring of germs of polynomial functions on $\mathbb{R}^n$.

c. The ring of germs of functions of differentiability class C^i, $i \geqslant 0$.

d. The ring of germs of continuous functions.

e. The ring of germs of real analytic functions on $\mathbb{R}^n$.

Exercise 4.17. Show that a ring with a maximal ideal I is local iff each element $r \notin I$ is invertible.

Definition 4.10. Zero divisors in a ring are non-zero elements r_1, r_2, saisfying $r_1 r_2 = 0$. **Nilpotent** is $r \in R$ such that $r^n = 0$ for some n.

[1] You are not required to prove Zorn's lemma in this exercise.

Exercise 4.18. Find whether the following rings have zero divisors.

a. The ring of germs of smooth functions.

b. The ring of germs of polynomial functions.

c. The ring of germs of continuous functions.

Definition 4.11. A continuous function f on $\mathbb{R}^n$ is called **piecewise polynomial** if $\mathbb{R}^n$ is represented as a union of polyhedra, and on each of these polyhedra, f is polynomial.

Exercise 4.19. Let $\mathcal{F}$ – a sheaf of piecewise polynomial functions on $\mathbb{R}$, S – a ring of its germs at 0.

a. Find out whether S is a local ring.

b. Show that S is isomorphic to $\mathbb{R}[t_1, t_2]/(t_1 t_2 = 0)$.

Exercise 4.20 (!). Let R be a local ring, $\mathfrak{m}$ its maximal ideal, and $K(R) := \bigcap_i \mathfrak{m}^i$. Prove that it is an ideal. Find whether this ideal is zero for

a. The ring of germs of smooth functions.

b. The ring of germs of real analytic functions.

c. The ring of germs of continuous functions.

Exercise 4.21 (*). Let $R = k[t_1, ..., t_n]$ be a ring of polynomials over a field, and $I \subset R$ an ideal.[2] Prove that $\bigcap_i I^i = 0$.

Exercise 4.22. Let R be a ring of germs of smooth functions in x, $\mathfrak{m}$ its maximal ideal, and $K(R) := \bigcap_i \mathfrak{m}^i$. Prove that for all $f \in K(R)$, all derivatives of f in zero (of any order) vanish.

Exercise 4.23. Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$, and f a function with all derivatives of any order vanishing. Show that $\frac{f}{(\sum_i x_i^2)^p}$ is continuous for any $p > 0$.

Exercise 4.24 (!). Under assumptions of the previous exercise, prove that the function $\frac{f}{\sum_i x_i^2}$ is smooth.

Exercise 4.25 (!). Let R be a ring of germs of smooth functions in $x \in \mathbb{R}^n$, $K(R) := \bigcap_i \mathfrak{m}^i$ the ideal defined above. Prove that $K(R)$ is an ideal of functions with vanishing derivatives of any order at x.

[2] The ideals in R are tacitly assumed to be $\neq R$.

Hint. Use the previous exercise.

Exercise 4.26 (*). Let $R/K(R)$ be the ring defined above.

a. Are there non-zero nilpotents in $R/K(R)$?

b. Are there zero divisors in $R/K(R)$?

4.3 Soft sheaves

Definition 4.12. Let $(M, \mathcal{F})$ be a topological space ringed by a sheaf of functions, and $X \subset M$ its subset. Consider a diagram indexed by open subsets $U_i \subset M$ containing X, with arrows corresponding to inclusions $U_j \subset U_i$, and associate with each U_i the corresponding section space $\mathcal{F}(U_i)$. A direct limit of this diagram is called **the ring of germs of $\mathcal{F}$ in X,** and denoted as $\mathcal{F}(X)$.

Exercise 4.27 (*). Let $(M, C^\infty M)$ be a manifold ringed by a sheaf of smooth functions, and $X \subset M$. Suppose that the space of germs of $C^\infty M$ in X is a local ring. Prove that X is a point.

Definition 4.13. A ring of functions $\mathcal{F}$ on M is called **soft** if for any closed subset $X \subset M$, the natural map from the space of global sections $\mathcal{F}(M)$ to the space of germs $\mathcal{F}(X)$ is surjective.

Exercise 4.28. Show that the sheaf of real analytic functions on $\mathbb{R}^n$ is not soft.

Exercise 4.29. Show that a constant sheaf on a manifold is not soft.[3]

Exercise 4.30. Find a topological space M and a sheaf of functions $\mathcal{F}$ on it such that the restriction map from $\mathcal{F}(M)$ to the space of germs of $\mathcal{F}$ in a point is always surjective, but the sheaf $\mathcal{F}$ is not soft.

Exercise 4.31. Let $N, N' \subset M$ be two closed subsets of a metric space, $N \cap N' = \emptyset$. Prove that there exist non-intersecting neighbourhoods $U \supset N$, $U' \supset N'$.

Exercise 4.32 (!). Let M be a manifold admitting a partition of unity, $N \subset M$ a closed subset, and $U \supset N$ its neighbourhood. Prove that M has a locally finite cover $\{U_i\}$ such that all U_i which intersect N are contained in U.

Hint. Prove that M admits a metric, and use the previous exercise.

Definition 4.14. Support of a function f is the set of all points where $f \neq 0$. A function is called **supported in** U if its support is contained in U.

[3]All manifolds are tacitly assumed to be of positive dimension.

Exercise 4.33. Let $U \subset M$ be an open subset of a manifold, $U' \Subset M$ an open subset satisfying $\bar{U}' \subset U$, and f a smooth function on U with support in U'. Prove that f can be extended to a smooth function on M.

Exercise 4.34 (*). Let M be a manifold admitting a partition of unity. Prove that the sheaf of smooth functions on M is soft.

Hint. Given a smooth function f on $U \supset N$, find a cover $\{U_i\}$, $i \in I$ as in previous exercise, and let $\{\psi_i\}$ be a subordinate partition of unity. Let $A \subset I$ be the set of indices $\alpha \in I$ such that $U_\alpha \cap N \neq 0$. Prove that the function $f' := \sum_{\alpha \in A} \psi_\alpha f$ is supported in $U' \Subset U$, can be extended smoothly to the whole M, and equal f on N.

Definition 4.15. Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$ on M. **Support** of f is the set of all points $x \in M$ such that there is no neighbourhood $U \ni x$ such that $f\big|_U = 0$.

Exercise 4.35. Prove that support of any section is closed.

Definition 4.16. A sheaf $\mathcal{F}$ on M is called **fine** if for any locally finite cover $\{U_\alpha\}$ of an open set $U \subset M$ indexed by $\alpha \in I$ and any section $f \in \mathcal{F}(U)$ there exists a collection of sections $f_\alpha \in \mathcal{F}(U)$ indexed by the same set I such that a support of any f_α is contained in U_α, and $\sum_I f_\alpha = f$.

Remark 4.5. Essentially the fine sheaves are sheaves which admit partition of unity.

Exercise 4.36 (*). Let M be a smooth manifold. Prove that the sheaf of smooth functions is fine.

Exercise 4.37 (*). Let M be a smooth manifold. Prove that the sheaf of smooth functions is soft.

Hint. Use the previous exercise.

Exercise 4.38 (**). Let M be a metrizable topological space. Prove that the sheaf of continuous functions is fine.

Exercise 4.39 (**). Let M be a metrizable topological space. Find a soft sheaf on M which is not fine.

Exercise 4.40 (**). Let $\mathcal{F}$ be a soft sheaf of functions, with the rings of germs local at all points. Prove that $\mathcal{F}$ is fine, or find a counterexample.

Exercise 4.41 (**). Let M be a metrizable topological space. Prove that any fine sheaf on M is soft.