Geometry 5: Vector bundles and sheaves

Rules: Exam problems would be similar to ones marked with ! sign. It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

5.1 Sheaves of modules.

Remark 5.1. Now I will give a new definition of a sheaf. The old definition ("sheaf of functions") becomes a special case of this one.

Definition 5.1. Let M be a topological space. **A sheaf** $\mathcal{F}$ on M is a collection of vector spaces $\mathcal{F}(U)$ defined for each open subset $U \subset M$, with the **restriction maps**, which are linear homomorphisms $\mathcal{F}(U) \stackrel{\phi_{U,U'}}{\longrightarrow} \mathcal{F}(U')$, defined for each $U' \subset U$, and satisfying the following conditions.

(A) Composition of restrictions is again a restriction: for any open subsets $U_1 \subset U_2 \subset U_3$, the corresponding restriction maps

$$\mathcal{F}(U_1) \stackrel{\phi_{U_1,U_2}}{\longrightarrow} \mathcal{F}(U_2) \stackrel{\phi_{U_2,U_3}}{\longrightarrow} \mathcal{F}(U_3)$$

give $\phi_{U_1,U_2} \circ \phi_{U_2,U_3} = \phi_{U_1,U_3}$.[1]

(B) Let $U \subset M$ be an open subset, and $\{U_i\}$ a cover of U. For any $f \in \mathcal{F}(U)$ such that all restrictions of f to U_i vanish, one has $f = 0$.

(C) Let $U \subset M$ be an open subset, and $\{U_i\}$ a cover of U. Consider a collection $f_i \in \mathcal{F}(U_i)$ of sections, defined for each U_i, and satisfying

$$f_i\Big|_{U_i\cap U_j} = f_j\Big|_{U_i\cap U_j}$$

for each U_i, U_j. Then there exists $f \in \mathcal{F}(U)$ such that the restriction of f to U_i is f_i.

The space $\mathcal{F}(U)$ is called **the space of sections of the sheaf $\mathcal{F}$ on U**. The restriction maps are often denoted $f \longrightarrow f\Big|_U$

Remark 5.2. For a sheaf of functions, the conditions (A) and (B) are satisfied automatically.

Exercise 5.1. Let M be a topological space equipped with a presheaf $\mathcal{F}$. Prove that the conditions (B) an (C) are equivalent to exactness of the following sequence.

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i\neq j} \mathcal{F}(U_i \cap U_j)$$

for any open $U \subset M$ an open subset, and any cover $\{U_i\}$ of U.

[1] If (A) is satisfied, $\mathcal{F}$ is called **a presheaf**.

Exercise 5.2. Let $f, g \in C^\infty M$ be functions which are equal on an open subset $U \subset M$, and $D \in \operatorname{Der}_{\mathbb{R}}(C^\infty M)$ a derivation on a ring of smooth functions. Prove that $D(f)\Big|_U = D(g)\Big|_U$.

Definition 5.2. Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

Exercise 5.3. Let $U \Subset V$ be open subsets in a smooth metrizable manifold. Prove that there exists a smooth function $\Phi_{U,V} \in C^\infty M$ supported on V and equal to 1 on U.

Exercise 5.4. Let $D \in \operatorname{Der}_{\mathbb{R}} C^\infty M$ be a derivation, and $U \Subset V$ open subsets in M. Given $f \in \mathbb{C}^\infty V$, define $D(f)\Big|_U$ using the formula $D(f)\Big|_U = D(\Phi_{U,V} \cdot f)$. Prove that $D(f)\Big|_U$ satisfies the Leibnitz rule, and is independent from the choice of $\Phi_{U,V}$.

Exercise 5.5 (!). Let $D \in \operatorname{Der}_{\mathbb{R}} C^\infty M$ be a derivation, and $V \subset M$ an open subset in M.

a. Prove that D can be extended to a derivation $D_V \in \operatorname{Der}_{\mathbb{R}} C^\infty V$, in such a way that $D_V\left(f\Big|_V\right) = D(f)\Big|_V$.

b. Prove that such an extension is unique.

Hint. Use the previous exercise.

Exercise 5.6 (!). Show that this construction makes $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ into a sheaf of modules over $C^\infty M$.

Definition 5.3. **A sheaf homomorphism** $\psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is a collection of homomorphisms

$$\psi_U : \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U),$$

defined for each $U \subset M$, and commuting with the restriction maps. **A sheaf isomorphism** is a homomorphism $\Psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi : \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such thate $\Phi \circ \Psi = \mathsf{Id}$ and $\Psi \circ \Phi = \mathsf{Id}$.

Exercise 5.7. Let $\psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be a sheaf homomorphism.

a. Show that $U \longrightarrow \ker \psi_U$ and $U \longrightarrow \operatorname{coker} \psi_U$ are presheaves.

b. Prove that $U \longrightarrow \ker \psi_U$ is a sheaf (it is called **the kernel** of a homomorphism ψ).

c. (*) Prove that $U \longrightarrow \operatorname{coker} \psi_U$ is not always a sheaf (find a counterexample).

Definition 5.4. **A subsheaf** $\mathcal{F}' \subset \mathcal{F}$ is a sheaf associating to each $U \subset M$ a subspace $\mathcal{F}'(U) \subset \mathcal{F}(U)$.

Exercise 5.8. Find a non-zero sheaf $\mathcal{F}$ on M such that $\mathcal{F}(M) = 0$.

Remark 5.3. Let $A: \phi \longrightarrow B$ be a ring homomorphism, and V a B-module. Then V is equipped with a natural A-module structure: $av := \phi(a)v$.

Definition 5.5. A sheaf of rings on a manifold M is a sheaf $\mathcal{F}$ with all the spaces $\mathcal{F}(U)$ equipped with a ring structure, and all restriction maps ring homomorphisms.

Definition 5.6. Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \stackrel{\phi_{U,U'}}{\longrightarrow} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use Remark 5.3 to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

Exercise 5.9. Let $\mathcal{F}_1$ be a sheaf of rings and $\mathcal{F}$ its subsheaf. Prove that $\mathcal{F}$ is a sheaf of modules over $\mathcal{F}$.

Definition 5.7. The space of germs of a sheaf $\mathcal{F}$ at $x \in M$ is the limit $\varinjlim \mathcal{F}(U)$, where U is the set of all neighbourhoods of x, and the maps are restriction maps.

Exercise 5.10. Let $\mathcal{F}$ be a ring sheaf on M. Prove that the space of germs of a sheaf of $\mathcal{F}$-modules is a module over the ring of germs of $\mathcal{F}$ in x.

Exercise 5.11. Let $\mathcal{B}$ be a sheaf with all germs equal 0. Prove that $\mathcal{B} = 0$.

Exercise 5.12 (*). Find a sheaf $\mathcal{F}$ on M with all germs non-zero, and $\mathcal{F}(M)$ zero.

Definition 5.8. A sheaf is called **globally generated** if for any $x \in M$, the natural restriction map $\mathcal{F}(M) \longrightarrow \mathcal{F}_x$ from the space of global sections to the space of germs is surjective.

Exercise 5.13 (*). Let $\mathcal{F}$ be a globally generated sheaf on M, and $U \subset M$ an open subset. Prove that the map $\mathcal{F}(M) \longrightarrow \mathcal{F}(U)$ is always surjective, or find a counterexample.

Exercise 5.14 (*). Let M be a smooth, metrizable manifold, and $\mathcal{F}$ be a sheaf of modules over $C^\infty(M)$. Prove that $\mathcal{F}$ is globally generated.

Definition 5.9. A free sheaf of modules $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$. A sheaf of $\mathcal{F}$-modules is **non-free** if it is not isomorphic to a free sheaf.

Exercise 5.15 (!). Find a subsheaf of modules in $C^\infty M$ which is non-free in the sense of this definition.

Definition 5.10. Locally free sheaf of modules over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}\Big|_U$ is free.

Exercise 5.16. Prove that a sheaf of $C^\infty M$-modules $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ is locally free, for each manifold M.

Exercise 5.17. Prove that $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ is a free sheaf for the following manifolds.

a. $M = \mathbb{R}$

b. $M = S^1$ (a circle)

c. $M = \mathbb{R}^2/\mathbb{Z}^2$ (a torus)

d. (*) $M = S^3$ (a three-dimensional sphere)

Exercise 5.18 (*). Find a manifold for which the sheaf $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ is not free.

Definition 5.11. A vector bundle on a ringed space $(M, \mathcal{F})$ is a locally free sheaf of $\mathcal{F}$-modules.

Definition 5.12. The sheaf of C^∞-modules $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ is called **a tangent bundle** to M.

Exercise 5.19 (!). Let B be a vector bundle on a manifold $(M, C^\infty M)$. Prove that B is globally generated (as a sheaf).

Exercise 5.20 (**). Let B_1, B_2 be vector bundles on (M, C^∞) such that the spaces of sections $B_1(M)$ and $B_2(M)$ are isomorphic as $C^\infty(M)$-modules. Prove that the bundles B_1 and B_2 are isomorphic.

Exercise 5.21 (!). Let $\mathcal{F}$ be a locally free sheaf of $C^\infty M$-modules. Prove that $\mathcal{F}$ is soft.

Exercise 5.22 (**). Let $\mathcal{F}$ be a sheaf of $C^\infty M$-modules. Prove that $\mathcal{F}$ is soft, or find a counterexample.

Definition 5.13. Let $\mathcal{F}$ be a sheaf of $C^\infty M$-modules, and $\mathcal{F}_x$ its germ in x. Denote the quotient $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ by $\mathcal{F}\Big|_x$. This space is called **the fiber** of $\mathcal{F}$ in x. A morphism of sheaves induces a linear map on each of its fibers.

Exercise 5.23 (**). Let $\mathcal{F}$ be a sheaf of $C^\infty M$-modules such that all its fibers $\mathcal{F}\Big|_x$ vanish. Prove that $\mathcal{F}$ is zero, or find a counterexample.