Geometry 6: Smooth fibrations

Rules: Exam problems would be similar to ones marked with ! sign. It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

6.1 Locally trivial smooth fibrations

Definition 6.1. Let $M \xrightarrow{\phi} N$ be a differentiable map of smooth manifolds. A **critical point** of ϕ is a point $m \in M$ where its differential $d\phi$ has rank less than maximal possible: $r < \min(\dim M, \dim N)$.

Exercise 6.1. Let $M \xrightarrow{\phi} N$ be a map without critical points, $\dim M > \dim N$, and $X \subset N$ a smooth submanifold. Prove that $\phi^{-1}(X)$ is a smooth submanifold in M.

Hint. Use the implicit function theorem.

Definition 6.2. A trivial smooth fibration is a projection $N \times U \longrightarrow U$, where N and U are smooth manifolds.

Definition 6.3. A surjective smooth map of manifolds $M \xrightarrow{\phi} N$ is called **a locally trivial smooth fibration** if each $x \in N$ has a neighbourhood $U \ni x$ such that the projection $\phi^{-1}(U) \longrightarrow U$ is a trivial smooth fibration.

Remark 6.1. Let $M \xrightarrow{\phi} N$ be a locally trivial smooth fibration, and $U \subset N$ an open subset. The map $\phi^{-1}(U) \longrightarrow U$ is called **restriction of the locally trivial fibration to $U \subset N$.**

Exercise 6.2. Show that any locally trivial fibration is a map without critical points.

Exercise 6.3. Consider a 3-dimensional sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$, and let $\pi: S^3 \longrightarrow \mathbb{C}P^1$ be a projection induced by the tautological map $\mathbb{C}^2\backslash 0 \longrightarrow \mathbb{C}P^1$. Show that it is a locally trivial fibration with fiber S^1.

Remark 6.2. This map is called **the Hopf fibration**.

Exercise 6.4. Let $\pi: S^3 \longrightarrow \mathbb{C}P^1$ be a Hopf fibration and $\mathbb{C} = \mathbb{C}P^1\backslash\{0\} \hookrightarrow \mathbb{C}P^1$ the standard embedding. Prove that $\pi^{-1}(\mathbb{C}P^1\backslash\{0\})$ is homeomorphic to $S^1 \times \mathbb{R}^2$.

Exercise 6.5. Prove that Hopf fibration is not a trivial fibration.

Exercise 6.6 (**). Prove **Ehresmann theorem**: any surjective, smooth map of compact manifolds without critical points is a locally trivial fibration.

Exercise 6.7. a. Construct a surjective map $S^{2n+1} \longrightarrow \mathbb{C}P^n$ without critical points.

b. (**) Prove that this is a locally trivial, but non-trivial fibration.

Hint. Generalize the construction of Hopf fibration.

Exercise 6.8 (**). Construct a locally trivial smooth fibration $S^7 \longrightarrow S^4$. Prove that it is non-trivial.

Definition 6.4. Let $M_1 \xrightarrow{\pi_1} N$ and $M_2 \xrightarrow{\pi_2} N$ be continuous maps of topological spaces, and $\Delta \subset N \times N$ a diagonal. Let $\pi_1 \times \pi_2 : M_1 \times M_2 \longrightarrow N \times N$ be a natural projection. Define $M_1 \times_N M_2 := (\pi_1 \times \pi_2)^{-1}(\Delta)$. The space $M_1 \times_N M_2$ is called **a fibered product**, or **fiber product of M_1 and M_2 over N.**

Exercise 6.9 (!). Let $M_1 \xrightarrow{\pi_1} N$ and $N_2 \xrightarrow{\pi_2} N$ be locally trivial smooth fibrations with fibers F_1 and F_2. Prove that the natural map $M_1 \times_N M_2 \longrightarrow N$ is a locally trivial fibration with fiber $F_1 \times F_2$.

Exercise 6.10. Represent a Moebius strip as a smooth fibration $M \xrightarrow{\pi} S^1$ with fiber $]0,1[$. Prove that $M \times_{S^1} M$ is homeomorphic to $S^1 \times]0,1[\times]0,1[$.

Exercise 6.11 (*). Let $\pi: S^3 \longrightarrow \mathbb{C}P^1$ be a Hopf fibration. Prove that $S^3 \times_{\mathbb{C}P^1} S^3$ is homeomorphic to $S^3 \times S^1$.

6.2 Groups and fiber products

Definition 6.5. A topological group is a topological space equipped with the group operations (product and taking inverse) which are continuous and satisfy the group axioms.

Exercise 6.12. Let G be a subgroup of the group of matrices, with natural topology. Prove that it is a topological group.

Exercise 6.13. Construct a structure of topological group on S^3.

Exercise 6.14 (*). Can an even-dimensional sphere be a topological group?

Exercise 6.15 (*). Can a bouquet of two circles be a topological group?

Definition 6.6. Let $M \xrightarrow{f} N$, $M' \xrightarrow{f'} N$ be continuous maps (morphisms) of topological spaces. A morphism $M \xrightarrow{\psi} M'$ is called **a morphism over** N, if the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M' \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ N & \xrightarrow{\mathsf{Id}} & N \end{array}$$

Definition 6.7. Let $B \xrightarrow{\pi} M$ be a continuous map, and $B \times_M B \xrightarrow{\Psi} M$ - a morphism over M. This morphism is called **associative multiplication** if it is associative on the fibers of π, that is, satisfies $\Psi(a, \Psi(b,c)) = \Psi(\Psi(a,b),c)$ for every triple a,b,c in the same fiber. A section $M \xrightarrow{e} B$ is called **the unit** if the maps

$$B \xrightarrow{\mathsf{Id}_B \times e} B \times_M B \xrightarrow{\Psi} B$$

and

$$B \xrightarrow{e \times \mathsf{Id}_B} B \times_M B \xrightarrow{\Psi} B$$

are equal to Id_B. A morphism $\nu: B \longrightarrow B$ over M is called **group inverse** if each of the maps

$$B \xrightarrow{\Delta} B \times_M B \xrightarrow{\mathsf{Id}_B \times \nu} B \times_M B \xrightarrow{\Psi} B$$

and

$$B \xrightarrow{\Delta} B \times_M B \xrightarrow{\nu \times \mathsf{Id}_B} B \times_M B \xrightarrow{\Psi} B$$

is a constant map, mapping b to $e(\pi(b))$. A map $B \xrightarrow{\pi} M$ equipped with associative multiplication, unit and group inverse is called **a topological group over** M.

Exercise 6.16. Let $B \xrightarrow{\pi} M$ be a topological group over M. Show that all fibers of π ara topological groups.

Exercise 6.17. Let $G \times M \longrightarrow M$ be a trivial fibration. Assume that G is equipped with a set of continuous group operations, indexed by $m \in M$ and continuously depending on m (that is, the corresponding maps, say, $G \times G \times M \longrightarrow G$ are continuous). Prove that this date gives a structure of topological group over M on $G \times M$.

Exercise 6.18. Let B be a topological group over M. Consider the space of continuous sections $M \longrightarrow B$. Prove that it is a group.

6.3 Vector bundles and smooth fibrations

Exercise 6.19. Let G be an abelian group, and k a field. Suppose that for each non-zero $\lambda \in k$ there exists an automorphism $\phi_\lambda : G \longrightarrow G$, such that $\phi_\lambda \circ \phi_{\lambda'} = \phi_{\lambda\lambda'}$, and $\phi_{\lambda+\lambda'}(g) = \phi_\lambda(g) + \phi_{\lambda'}(g)$. Show that G is a vector space over k. Show that all vector spaces can be obtained this way.

Definition 6.8. Let $k = \mathbb{R}$ or $\mathbb{C}$. An abelian topological group $B \xrightarrow{\pi} M$ over M is called **relative vector space over** M if for each continuous k-valued function f there exists a continuous automorphism $\phi_f : B \longrightarrow B$ of a group B over M satisfying assumptions of Exercise 6.19, which makes each fiber $\pi^{-1}(b)$ into a vector space in such a way that ϕ_f acts on $\pi^{-1}(b)$ as a multiplication by $\phi_f(b)$.

Exercise 6.20. Let $B \xrightarrow{\pi} M$ be a relative vector space over M, $U \subset M$ an open subset, and $\mathcal{B}(U)$ the space of sections of a map $\pi^{-1}(U) \xrightarrow{\pi} U$.

a. Show that $\mathcal{B}(U)$ is a vector space.

b. Prove that $\mathcal{B}(U)$ defines a sheaf of modules over a sheaf $C^0(M)$ of continuous functions.

Exercise 6.21. Let $S \subset \mathbb{R}^n$ be a subset (not necessarily a smooth submanifold), $s \in S$ a point, and $v \in T_s\mathbb{R}^n$ a vector. We sat that v belongs to a **tangent cone** C_sS if the distance from S to a point $s + tv$ converges to 0 as $t \to 0$ faster than linearly:

$$\lim_{t \longrightarrow 0} \frac{d(S, s+tv)}{t} \longrightarrow 0.$$

a. (!) Let T_sS be a space generated by C_sS. Show that the set TS of all pairs $(s, v), s \in S, v \in T_sS$ is a relative vector space over S.[1]

b. (!) Find CS for set $S \subset \mathbb{R}^3$ of zeros of a polynomial $x^2 + y^2 - z^2$.

c. (!) Show that in this situation, $CS \longrightarrow S$ is not a locally trivial smooth fibration.

Definition 6.9. Let $B \longrightarrow M$ be a smooth locally trivial fibration with fiber $\mathbb{R}^n$. Assume that B is equipped with a structure of relative vector space over M, and all the maps used in the definition of a relative vector space are smooth. Then B is called **a total space of a vector bundle.**

Exercise 6.22. Let $B \longrightarrow M$ be a relative vector space over M, and $\mathcal{F}$ the corresponding sheaf of sections. Prove that it is a locally free sheaf of $C^\infty M$-modules.

[1] The space CS is called **a tangent cone to** S.

Definition 6.10. Recall that **a vector bundle** is a locally free sheaf of modules over $C^\infty M$. A vector bundle is called **trivial** if it is isomorphic to $C^\infty M^n$.

Definition 6.11. Let $\mathcal{B}$ be an n-dimensional vector bundle on M, $x \in M$ a point, $\mathcal{B}_x$ the space of germs of $\mathcal{B}$ in x, and $\mathfrak{m}_x \subset C^\infty_x M$ the maximal ideal in the ring of germs $C^\infty_x M$ of smooth functions. Define **the fiber** of $\mathcal{B}$ in x as a quotient $\mathcal{B}_x/\mathfrak{m}_x\mathcal{B}_x$. A fiber of a vector bundle is denoted $\mathcal{B}\big|_x$.

Exercise 6.23. Show that a fiber of an n-dimensional bundle is an n-dimensional vector space.

Exercise 6.24. Let $\mathcal{B} = C^\infty M^n$ be a trivial n-dimensional bundle on M, and $b \in \mathcal{B}\big|_x$ a point of a fiber, represented by a germ $\phi \in \mathcal{B}_x = C^\infty_m M^n$, $\phi = (f_1, ..., f_n)$. Consider a map from the set of all fibers $\mathcal{B}$ to $M \times \mathbb{R}^n$, mapping $(x, \phi = (f_1, ..., f_n))$ to $(f_1(x), ..., f_n(x))$. Prove that this map is bijective.

Definition 6.12. Let $\mathcal{B}$ be an n-dimensional vector bundle over M. Denote the set of all vectors in all fibers of $\mathcal{B}$ over all points of M by $\operatorname{Tot}\mathcal{B}$. Let $U \subset M$ be an open subset of M, with $\mathcal{B}\big|_U$ a trivial bundle. Using the local bijection $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ defined in Exercise 6.24, we consider topology on $\operatorname{Tot}\mathcal{B}$ induced by open subsets in $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ for all open subsets $U \subset M$ and all trivializations of $\mathcal{B}\big|_U$.

Exercise 6.25. Show that $\operatorname{Tot}\mathcal{B}$ with this topology is a locally trivial fibration over M, with fiber $\mathbb{R}^n$.

Exercise 6.26 (!). Show that $\operatorname{Tot}\mathcal{B}$ is equipped with a natural structure of a relative vector space over M, and the sheaf of smooth sections of $\operatorname{Tot}\mathcal{B} \longrightarrow M$ is isomorphic to $\mathcal{B}$.

Definition 6.13. Let $\mathcal{B}$ be a vector bundle on M. Then $B = \operatorname{Tot}\mathcal{B}$ is called **the total space of a vector bundle $\mathcal{B}$**.

Remark 6.3. In practice, "the total space of a vector bundle" is usually denoted by the same letter as the corresponding sheaf. Quite often, mathematicians don't even distinguish between these two notions.

Exercise 6.27. Let $M_1 \xrightarrow{\phi} M$ be a smooth map of manifolds, and $B \xrightarrow{\pi} M$ a total space of a vector bundle. Prove that $B \times_M M_1$ is a total space of a vector bundle on M_1.

Definition 6.14. This bundle is denoted $\phi^* B$, and called **inverse image**, or **a pullback** of B.

Exercise 6.28. Prove that the fiber $\phi^*(B)\big|_x$ is naturally identified with $B\Big|_{\phi(x)}$.

Exercise 6.29. Prove that a pullback of a trivial bundle is trivial.

Exercise 6.30. Let $M_1 \xrightarrow{\phi} M$ be a surjective, smooth map without critical points, and B a non-trivial bundle on M.

a. (*) Can the bundle $\phi^* B$ be trivial?

b. (*) Suppose that M_1 is compact. Can $\phi^* B$ be trivial?