# Geometry 7: Vector bundles

**Rules:** Exam problems would be similar to ones marked with ! sign. It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

## 7.1 Tensor product

**Definition 7.1.** Let $V, V'$ be $R$-modules, $W$ a free abelian group generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subgroup generated by combinations $rv \otimes v' - v \otimes rv'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$. Define **the tensor product** $V \otimes_R V'$ as a quotient group $W/W_1$.

**Exercise 7.1.** Show that $r \cdot v \otimes v' \mapsto (rv) \otimes v'$ defines an $R$-module structure on $V \otimes_R V'$.

**Exercise 7.2.** Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) = 0$.

**Exercise 7.3 (*).** Find a non-zero $R$-module $V$ such that $V \otimes_R V = 0$.

**Exercise 7.4.** Let $I_1, I_2$ be ideals in $R$. Prove that $(R/I_1) \otimes_R (R/I_2) = R/(I_1 + I_2)$, where $I_1 + I_2$ is an ideal generated by linear combinations $I_1, I_2$.

**Exercise 7.5.** Prove that a tensor product of free $R$-modules is free.

**Exercise 7.6.** Let $\mathcal{F}$ be a sheaf of rings, and $\mathcal{B}_1$ and $\mathcal{B}_2$ sheaves of locally free $(M, \mathcal{F})$-modules. Prove that

$$U \longrightarrow \mathcal{B}_1(U) \otimes_{\mathcal{F}(U)} \mathcal{B}_2(U)$$

is also a sheaf of modules.

**Exercise 7.7 (**).** Is the last statement true without the assumption of local triviality?

**Definition 7.2. Tensor product** of vector bundles is a tensor product of the corresponding sheaves of modules.

**Remark 7.1.** In a similar way one defines exterior powers and symmetric powers of a bundle.

**Exercise 7.8.** Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be locally free sheaves of $C^\infty M$-modules, and $\mathcal{B}_1 \otimes_{C^\infty M} \mathcal{B}_2$ their tensor product. Show that the fiber $\mathcal{B}_1 \otimes_{C^\infty M} \mathcal{B}_2$ in $x$ is naturally identified with a tensor product of the fibers:

$$\left(\mathcal{B}_1 \otimes_{C^\infty M} \mathcal{B}_2\right)\Big|_x \cong \mathcal{B}_1\Big|_x \otimes_{\mathbb{R}} \mathcal{B}_2\Big|_x .$$

**Exercise 7.9.** Let $V$ be an $R$-module, and $\operatorname{Hom}_R(V,R)$ the space of $R$-linear homomorphisms from $V$ to $R$. Prove that the action $r \cdot h(\dots) \mapsto rh(\dots)$ gives a structure of $R$-module on $\operatorname{Hom}_R(V,R)$. Prove that $\operatorname{Hom}_R(R^n,R)$ with $R$-module structure defined this way is isomorphic (non-canonically) to a free module $R^n$.

**Definition 7.3.** Let $V$ be an $R$-module. **A dual $R$-module** $V^*$ is $\operatorname{Hom}_R(V,R)$ with the $R$-module structure defined above.

**Exercise 7.10.** Consider $\mathbb{Q}/Z$ as a $Z$-module. Prove that $(\mathbb{Q}/Z)^* = 0$.

**Exercise 7.11.** Prove that $\operatorname{Hom}_{\mathbb{Z}}(Q,\mathbb{Z}) = 0$.

**Exercise 7.12 (*).** Let $R = C^\infty(\mathbb{R})_0$ be a ring of germs of smooth functions at 0, and $K$ an ideal of functions vanishing in 0 with all derivatives. Prove that $(R/K)^* := \operatorname{Hom}_R(R/K,R) = 0$, or disprove it.

**Exercise 7.13 (*).** Same question when $R = C^\infty(\mathbb{R}^n)_0$.

**Exercise 7.14 (!).** Let $\mathcal{B}$ be a vector bundle, that is, a locally free sheaf of $C^\infty M$-modules, and $\operatorname{Tot}\mathcal{B} \xrightarrow{\pi} M$ its total space. Define $\mathcal{B}^*(U)$ as a space of smooth functions on $\pi^{-1}(U)$ linear in the fibers of $\pi$.

a. Show that the natural restriction map $\mathcal{B}^*(U) \longrightarrow \mathcal{B}^*(V)$ defines a sheaf $\mathcal{B}^*$.

b. Show that this sheaf is locally trivial.

c. (!) Show that $\mathcal{B}^*(U)$ is a dual $C^\infty(U)$-module to $\mathcal{B}(U)$.

**Definition 7.4.** Let $\mathcal{B}$ be a vector bundle, and $\mathcal{B}^*$ a locally trivial sheaf of $C^\infty M$-modules defined above. It is called **the dual bundle** to $\mathcal{B}$.

**Exercise 7.15.** Prove that the fiber $\mathcal{B}^*\Big|_x$ is a vector space dual to $\mathcal{B}\Big|_x$.

**Exercise 7.16.** Let $\mathcal{B}$ be a non-trivial vector bundle. Prove that $\mathcal{B}^*$ is also non-trivial.

**Definition 7.5. Bilinear form** on a bundle $\mathcal{B}$ is a section of $(\mathcal{B}\otimes\mathcal{B})^*$. A symmetric bilinear form on $\mathcal{B}$ is called **positive definite** if it gives a positive definite form on all fibers of $\mathcal{B}$. Symmetric positive definite form is also called **a metric**. A skew-symmetric bilinear form on $\mathcal{B}$ is called **non-degenerate** if it is non-degenerate on all fibers of $\mathcal{B}$.

**Exercise 7.17 (!).** Let $\mathcal{B}$ be a vector bundle on a metrizable manifold $M$. Prove that $\mathcal{B}$ admits a metric.

**Hint.** Construct the metric locally, and use partition of unity.

**Exercise 7.18.** Construct a 2-dimensional vector bundle which does not admit a non-degenerate skew-symmetric bilinear form.

**Exercise 7.19 (**).** Let $M$ be a simply connected manifold, and $B$ a $2n$-dimensional vector bundle. Prove that $B$ admits a non-degenerate skew-symmetric bilinear form, or find a counterexample.

**Exercise 7.20 (*).** Find a non-trivial 3-dimensional bundle $\mathcal{B}$ such that its exterior square $\Lambda^2\mathcal{B}$ is trivial.

**Exercise 7.21 (*).** Find a 2-dimensional bundle which does not admit a non-degenerate bilinear symmetric form of signature $(1, 1)$.

## 7.2 Smooth morphisms of vector bundles and subbundles

**Definition 7.6.** Let $\mathcal{B}, \mathcal{B}'$ be sheaves on $M$. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \downarrow & & \downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

**Remark 7.2.** Morphisms of sheaves of modules are defined in the same way, but in this case the maps $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$ should be compatible with the module structure.

**Definition 7.7.** A sheaf morphism is called **injective** if it is injective on germs and **surjective**, if it is surjective on germs.

**Exercise 7.22.** Let $\mathcal{B} \xrightarrow{\phi} \mathcal{B}'$ be an injective morphism of sheaves on $M$. Prove that $\phi$ induces an injective map $\mathcal{B}(M) \longrightarrow \mathcal{B}'(M)$ on the spaces of global sections.

**Exercise 7.23 (*).** Find an example of a surjective sheaf morphism which is not surjective on global sections.

**Definition 7.8.** Let $\mathcal{B} \xrightarrow{\phi} \mathcal{B}'$ be a morphism of locally free sheaves of $C^\infty M$-modules. It is called **a smooth morphism**, or **a morphism of vector bundles** if on each of the germ spaces $\phi$ has free kernel and free cokernel.

**Definition 7.9.** Let $\mathcal{F}$ be a locally free sheaf of $C^\infty M$-modules, and $\mathcal{F}_x$ its space of germs in $x$. Denote the quotient $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ by $\mathcal{F}\Big|_x$. This space is called **the fiber** of $\mathcal{F}$ in $x$. A morphism of sheaves induces a linear map on each of its fibers.

**Exercise 7.24 (!).** Find an example of an injective morphism of locally free $C^\infty M$-modules which is not injective in some fiber.

**Exercise 7.25 (*).** Prove that a surjective morphism of locally free sheaves of $C^\infty M$-modules is a smooth morphism of vector bundles, in the sense of the above definition.

**Exercise 7.26.** Let $\mathcal{B} \longrightarrow \mathcal{B}_1$ be a smooth morphism of vector bundles on $M$.

a. Prove that the corresponding map $\Psi$ of total spaces is a homomorphism of relative vector spaces over $M$.

b. Prove that $\Psi$ has no critical points.

**Definition 7.10.** **A subbundle** $\mathcal{B}_1 \subset \mathcal{B}$ is an image of an injective morphism of vector bundles.

**Exercise 7.27.** Let $\mathcal{B}_1 \subset \mathcal{B}$ be a subbundle. Prove that the quotient $\mathcal{B}/\mathcal{B}_1$ is also a vector bundle.

**Exercise 7.28 (!).** Let $\mathcal{B}_1 \xrightarrow{\phi} \mathcal{B}_2$ be a morphism of vector bundles. Prove that the image of $\phi$ is a subbundle in $\mathcal{B}_2$, and its kernel is a subbundle in $\mathcal{B}_1$.

**Definition 7.11.** **Direct sum** of vector bundles is a direct sum of corresponding sheaves.

**Exercise 7.29.** Prove that a total space of a direct sum of vector bundles $\mathcal{B} \oplus \mathcal{B}'$ is homeomorphic to $\operatorname{Tot} \mathcal{B} \times_M \operatorname{Tot} \mathcal{B}'$.

**Exercise 7.30.** Let $\mathcal{B}$ be a vector bundle equipped with a metric (that is, a positive definite symmetric form), and $\mathcal{B}_1 \subset \mathcal{B}$ a subbundle. Consider a subset $\operatorname{Tot} \mathcal{B}_1^\perp \subset \operatorname{Tot} \mathcal{B}$, consisting of all $v \in \mathcal{B}\big|_x$ orthogonal to $\mathcal{B}_1\big|_x \subset \mathcal{B}\big|_x$. Prove that $\operatorname{Tot} \mathcal{B}_1^\perp$ is a total space of a subbundle, denoted as $\mathcal{B}_1^\perp \subset \mathcal{B}$.

**Definition 7.12.** A subbundle $\mathcal{B}_1^\perp \subset \mathcal{B}$ is called **orthogonal complement** of $\mathcal{B}$ to $\mathcal{B}_1 \subset \mathcal{B}$.

**Exercise 7.31.** Let $\mathcal{B}_1 \subset \mathcal{B}$ be a sub-bundle. Prove that $\mathcal{B}$ is isomorphic to a direct sum of $\mathcal{B}_1$ and another bundle.

**Hint.** Find a metric on $\mathcal{B}$ and use the previous exercise.

**Remark 7.3.** In this situation, it is said that $\mathcal{B}_1$ **is a direct summand of** $\mathcal{B}$.