# Geometry 9: de Rham differential

**Rules:** Exam problems would be similar to ones marked with ! sign. It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

## 9.1 Kähler differentials

**Definition 9.1.** Let $R$ be a ring over a field $k$, and $V$ an $R$-module. A $k$-linear map $D: R \longrightarrow V$ is called **a derivation** if it satisfies **the Leibnitz identity** $D(ab) = aD(b) + bD(a)$. The space of derivations from $R$ to $V$ is denoted $\operatorname{Der}_k(R, V)$.

**Exercise 9.1.** Consider an action of $R$ on $\operatorname{Der}_k(R, V)$, with $rd$ acting as $a \longrightarrow rd(a)$. Prove that this defines a structure of $R$-module on $\operatorname{Der}_k(R, V)$.

**Exercise 9.2.** Let $[K : k]$ be a finite extension of a field of characteristic 0, and $V$ a vector space over $K$. Prove that $\operatorname{Der}_k(K, V) = 0$.

**Exercise 9.3 (!).** Let $M$ be a smooth manifold, $x \in M$ a point, $R = C^\infty M$, and $\mathfrak{m}_x \subset R$ the maximal ideal of $x$. Consider an $R$-module $V := R/\mathfrak{m}_x$. Find $\dim_{\mathbb{R}} \operatorname{Der}_R(R, V)$.

**Exercise 9.4 (**).** Let $R = C^0 M$ be a ring of continuous functions on a manifold $M$, and $V$ an $R$-module of dimension 1 over $\mathbb{R}$. Find a non-trivial derivation $\nu \in \operatorname{Der}_k(R, V)$, or prove that it does not exist.

**Definition 9.2.** Let $R$ be a ring over a field $k$. Define an $R$-module $\Omega^1_k R$, sometimes denoted simply as $\Omega^1 R$, with the following generators and relations. The generators of $\Omega^1_k R$ are indexed by elements of $R$; for each $a \in R$, the corresponding generator of $\Omega^1_k R$ is denoted $da$. Relations in $\Omega^1_k R$ are generated by expressions $d(ab) = adb + bda$, for all $a, b \in R$, and $d\lambda = 0$ for each $\lambda \in k$. Then $\Omega^1_k R$ is called **the module of Kähler differentials** of $R$.

**Exercise 9.5.** Prove that the natural map $R \longrightarrow \Omega^1_k R$, with $a \mapsto da$ is a derivation.

**Exercise 9.6.** Let $R$ be a quotient of $k[r_1, ..., r_k]$ by an ideal. Prove that $\Omega^1_k R$ is generated as an $R$-module by $dr_1, ..., dr_k$.

**Exercise 9.7 (!).** Prove that $\operatorname{Der}_k(R) = \operatorname{Hom}_R(\Omega^1_k R, R)$.

**Exercise 9.8.** Let $V$ be an $R$-module, and $D \in \operatorname{Der}_k(R, V)$ a derivation. Prove that there exists a unique $R$-module homomorphism $\phi_D: \ \Omega^1_k R \longrightarrow V$ making the following diagram commutative.

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[dr, "D"'] & \Omega^1_k R \arrow[d, "\phi_D"] \\ & V \end{tikzcd}$$

**Remark 9.1.** This property is often taken as a definition of $\Omega^1_k R$.

**Exercise 9.9 (!).** Let $R = k[t_1, ..., t_n]$ be a polynomial ring over a field of characteristic 0. Prove that $\Omega^1_k R$ is a free $R$-module generated by $dt_1, dt_2, ..., dt_n$.

**Exercise 9.10 (*).** Let $I \subset R$ be an ideal. Construct an exact sequence

$$I/I^2 \longrightarrow \Omega^1(R) \otimes_R R/I \longrightarrow \Omega^1(R/I) \longrightarrow 0.$$

**Exercise 9.11.** Let $R \xrightarrow{\phi} R'$ be a ring homomorphism. Consider $\Omega^1 R'$ as an $R$-module, using the action $r, a \longrightarrow \phi(r)a$.

a. Prove that there exists an $R$-module homomorphism $\Omega^1 R \longrightarrow \Omega^1 R'$, mapping $dr$ to $d\phi(r)$.

b. Prove that it is unique.

**Definition 9.3.** In this case, we say that the homomorphism $\Omega^1 R \longrightarrow \Omega^1 R'$ **is induced by** $\phi$.

**Exercise 9.12 (*).** Let $R$ be a ring of continuous functions on a manifold, and $\mathfrak{m}_x$ a maximal ideal of a point. Prove that $\mathfrak{m}_x \Omega^1 R = \Omega^1 R$.

## 9.2 Cotangent bundle

**Definition 9.4.** Let $A, B$ be $R$-modules, and $\nu: A \times B \longrightarrow R$ a bilinear pairing. It is called **non-degenerate** if for each $a \in A$ there exists $b \in B$ such that $\nu(a,b) \neq 0$, and for each $b \in B$ there exists $a \in A$ such that $\nu(a,b) \neq 0$

**Exercise 9.13.** Let $A, B$ be vector spaces over $k$, and $\nu: A \times B \longrightarrow k$ a non-degenerate pairing. Prove that $A$ is isomorphic to $B^*$, or find a counterexample

a. When $A, B$ are finite-dimensional.

b. When $A, B$ are infinite-dimensional.

**Definition 9.5.** Let $V$ be an $R$-module. A **dual $R$-module** $\operatorname{Hom}_R(V, R)$ is denoted $R^*$.

**Exercise 9.14.** Let $V$ be an $R$-module. Consider the natural pairing $V \times V^* \longrightarrow R$. Prove that it is non-degenerate, or find a counterexample, in the following cases:

a. when $R$ is a field, and $V$ a (possibly infinite-dimensional) vector space

b. (!) when the natural map $V \longrightarrow V^{**}$ is injective

c. when $V$ is a free $R$-module

d. (*) when $R$ is a ring which has no zero divisors.

**Exercise 9.15 (*).** Let $A, B$ be finitely-generated $R$-modules, and $\nu: A \times B \longrightarrow R$ a non-degenerate pairing. Prove that $A$ is isomorphic to $B^*$, or find a counterexample.

**Exercise 9.16 (!).** Let $A$ be a free, finitely generated $R$-module, and $\nu: A \times B \longrightarrow R$ a non-degenerate pairing. Prove that $B$ is also free, and isomorphic to $A^*$.

**Definition 9.6.** Let $A$, $B$ be finitely generated $R$-modules, and $\nu: A \times B \longrightarrow R$ a bilinear pairing. Define **the annihilator of $\nu$ in $B$** as a submodule consisting of all elements $b \in B$ for which the homomorphism $\nu(\cdot, b): A \longrightarrow R$ vanishes.

**Definition 9.7.** Let $M$ be a smooth manifold, $R := C^\infty M$ the ring of smooth finctions, and $\nu: \operatorname{Der}(R) \times \Omega^1 R \longrightarrow R$ the pairing constructed in Exercise 9.7. Consider its annihilator $K \subset \Omega^1 R$. Define **the cotangent bundle** as $\Lambda^1 M := \Omega^1 R/K$. For the purpose of this definition, $\Lambda^1 M$ is a $C^\infty M$-module.

**Exercise 9.17.** Let $R := C^\infty \mathbb{R}^n$, and $t_1, ..., t_n \in R$ be coordinate functions. Consider an element in $\Lambda^1 \mathbb{R}^n$, written as $P = \sum_{i=1}^n P_i dt_i$, let $Q = \sum_{i=1}^n Q_i \frac{d}{dt_i} \in \operatorname{Der}_k(R)$ – be a vector field, and $\nu: \operatorname{Der}(R) \times \Lambda^1 \mathbb{R}^n \longrightarrow R$ the natural pairing. Prove that $\nu(P, Q) = \sum_i P_i Q_i$.

**Exercise 9.18 (!).** In these assumptions, prove that $\Lambda^1 R$ is a free $\mathbb{R}$-module, generated by $dt_1, ..., dt_n$.

**Hint.** Prove that $\operatorname{Der}(R) = \operatorname{Hom}_R(\Omega^1 R, R)$, and $\operatorname{Der}(R)$ is a free $R$-module. Use exercise 9.16.

**Exercise 9.19.** Let $A, B$ be finitely-generated projective $R$-modules, and $\nu: A \times B \longrightarrow R$ a non-degenerate pairing. Prove that $B \cong A^*$.

**Exercise 9.20 (!).** Let $M$ be a smooth, metrizable manifold. Prove that

$$\Lambda^1 M = \operatorname{Hom}_{C^\infty M}(\operatorname{Der}(C^\infty M), C^\infty M).$$

**Hint.** Use the previous exercise and apply the Serre-Swan theorem.

**Exercise 9.21 (*).** Let $K$ be the kernel of the natural projection

$$\Omega^1(C^\infty M) \longrightarrow \Lambda^1 M.$$

Prove that $\mathfrak{m}_x K = K$ for each maximal ideal of a point $x \in M$.

**Exercise 9.22 (**).** Show that $K$ is non-empty.

## 9.3 De Rham algebra

**Definition 9.8.** Let $M$ be a smooth manifold. **A bundle of differential $i$-forms on $M$** is the bundle $\Lambda^i T^* M$ of antisymmetric $i$-forms on $TM$. It is denoted $\Lambda^i M$.

**Definition 9.9.** Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on $V$. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

**Exercise 9.23.** Let $\bigotimes_k T^* M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathsf{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define the "exterior multiplication" $\wedge: \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^* M$ obtained as their tensor multiplication. Prove that this operation is associative and satisfies $\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha$.

**Definition 9.10.** The algebra $\Lambda^* M := \oplus_i \Lambda^i M$ with the multiplicative structure defined above is called **the de Rham algebra** of a manifold.

**Exercise 9.24 (*).** Let $M$ be an oriented manifold. Prove that all bundles $\Lambda^i M$ are oriented, or find a counterexample.

**Exercise 9.25.** Prove that de Rham algebra is multiplicatively generated by $C^\infty M = \Lambda^0 M$ and $d(C^\infty) \subset \Lambda^1 M$.

**Exercise 9.26.** Prove that a derivation on an algebra is uniquely determined by its values on any set of multiplicative generators of this algebra.

**Definition 9.11. De Rham differential** $d: \Lambda^* M \longrightarrow \Lambda^{*+1} M$ is an $\mathbb{R}$-linear map satisfying the following conditions.

**(i)** For each $f \in \Lambda^0 M = C^\infty M$, $d(f) \in \Lambda^1 M$ is equal to the image of the Kähler differential $df \in \Omega^1 M$ in $\Lambda^1 M = \Omega^1 M/K$.

**(ii)** (Leibnitz rule) $d(a \wedge b) = da \wedge b + (-1)^j a \wedge db$ for any $a \in \Lambda^i M, b \in \Lambda^j M$.

**(iii)** $d^2 = 0$.

**Exercise 9.27 (!).** Prove that de Rham differential is defined uniquely by these axioms.

**Hint.** Use the previous exercise.

**Exercise 9.28.** Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha \in \Lambda^* \mathbb{R}^n$ a monomial obtained as a product of several $dt_i$,

$$\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k},$$

$i_1 < i_2 < ... < i_k$ (such a monomial is called **a coordinate monomial**).

a. Prove that $\Lambda^* \mathbb{R}^n$ is a trivial bundle, and coordinate monomials are free generators of $\Lambda^* \mathbb{R}^n$.

b. Show that the de Rham differential, if it exists, satisfies $d(f\alpha) = \sum_i \frac{df}{dt_i} dt_i \wedge \alpha$ for any $f \in C^\infty \mathbb{R}^n$.

c. Prove that this formula defines the de Rham differential on $\Lambda^* \mathbb{R}^n$ correctly.

**Exercise 9.29.** a. Prove that de Rham differential $d: \Lambda^* M \longrightarrow \Lambda^{*+1} M$ commutes with restrictions to open subsets.

b. Show that de Rham differential (if it exists) defines a sheaf morphism.

**Hint.** Use uniqueness of de Rham differential.

**Exercise 9.30 (!).** Prove that de Rham differential exists on any manifold.

**Hint.** Locally, de Rham differential is constructed in exercise 9.28. To go from local to global, use the previous exercise, and apply the sheaf axioms.

**Exercise 9.31 (*).** Let $R$ be a ring over a field, and $\Omega^i R := \Lambda^i_R \Omega^1 R$ an exterior algebra generated by Kähler differentials. Prove that there exists the de Rham differential $d: \Omega^* R \longrightarrow \Omega^{*+1} R$ satisfying the axioms above.