MATH-F-420: final exam

Rules: Every student receives from me a list of 10 exercises (chosen randomly), and has to solve as many of them as you can by January 17. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solutions, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes). Please contact me by email `verbit2000[]gmail.com` when you are ready. The final score for exam is $4k+4$, where k is the total number of points you got.

1 Manifolds and rings of smooth functions

Exercise 1.1. Consider **the Moebius strip** M as a quotient space of $\mathbb{R}\times[0,1]$ with opposite lines glued together with reverse orientation. Construct a closed submanifold in $\mathbb{R}^3$ which is diffeomorphic to a Moebius strip, or prove that it does not exist.

Exercise 1.2. Let M be an n-dimensional manifold. Construct a smooth, surjective map from M to the torus $(S^1)^n$.

Exercise 1.3 (2 points). Let R be a ring of continuous $\mathbb{R}$-valued functions on a compact topological space M, and $I\subset R$ an ideal. Let Z be the set of common zeros of all $f\in I$. Prove that Z is non-empty. Prove that I is ideal of all functions vanishing in Z or find a counterexample.

Exercise 1.4. Let $X,Y\subset M$ be closed subsets of a metric space M. Assume that $\inf_{x\in X, y\in Y} d(x,y)>1$. Prove that there exists a 1-Lipschitz function on M which is equal to 1 on X and 0 on Y.

Exercise 1.5 (2 points). Let M be a manifold, f a continuous function on M, and F the set of all smooth functions f on M satisfying $g\leqslant f$. Prove $f(x)=\sup_{g\in F} g(x)$.

Exercise 1.6. Let M be a connected smooth manifold, and G a group of all its diffeomorphisms with itself. Prove that G acts on M transitively on the set of $x_1, x_2, ..., x_n$ which are pairwise unequal.

2 Sheaves and smooth fibrations

Definition 2.1. A sheaf $\mathcal{B}$ is called **flasque** if any restriction map

$$\mathcal{B}(U)\longrightarrow\mathcal{B}(V)$$

is surjective.

Exercise 2.1. For a given sheaf $\mathcal{B}_1$, find a sheaf monomorphism $\mathcal{B}_1 \hookrightarrow \mathcal{B}$ to a flasque sheaf.

Definition 2.2. A sheaf $\mathcal{I}$ is called **injective** if for any sheaf morphism $\mathcal{B} \xrightarrow{\phi} \mathcal{I}$ and a monomorphism $\mathcal{B} \hookrightarrow \mathcal{B}'$, the map ϕ can be extended to a morphism $\mathcal{B}' \xrightarrow{\phi} \mathcal{I}$.

Exercise 2.2. Prove that any injective sheaf is flasque.

Exercise 2.3. Let B be a sheaf of modules over $C^\infty M$, and $\check{B}$ a sheaf given by $\check{B}(U) = B_c(U)^*$, where $B_c(U)$ is the space of sections with compact support. Prove that $\check{B}(U)$ is a flasque sheaf.

Definition 2.3. Recall that $U(n)$ is a group of all unitary automorphisms of $\mathbb{C}^n$, $SO(n)$ a group of all orthogonal, oriented automorphisms of $\mathbb{R}^n$. Also, $SU(n)$ is unitary automorphisms with determinant 1.

Exercise 2.4. Let G be $U(n)$ or $SO(n)$. Find a fibration with a total space G, base S^n (a sphere), and fiber G_0, which is also a Lie group. Find G_0 in each of these cases.

Exercise 2.5. Construct a fibration with total space $SU(3)$, base S^5 and fiber S^3.

3 Vector bundles

All vector bundles here are considered over reals (and not over complex numbers).

Definition 3.1. **Trivial bundle** over M is a direct sum of several copies of $C^\infty M$.

Exercise 3.1. Let $TS^2 \oplus \mathbb{R}$ be a direct sum of a tangent bundle TS^2 and a trivial 1-dimensional bundle. Is the bundle $TS^2 \oplus \mathbb{R}$ trivial?

Definition 3.2. A bundle B of rank (dimension) d is called **orientable**, or **oriented**, if its top exterior power $\Lambda^d B$ is a trivial bundle.

Exercise 3.2 (2 points). Let M be a smooth manifold, and TM the total space of its tangent bundle. Prove that TM is orientable, or find a counterexample.

Exercise 3.3 (2 points). Let M be an oriented manifold. Prove that all bundles $\Lambda^i M$ are orientable, or find a counterexample.

Exercise 3.4. Let M be a simply connected manifold. Prove that any real rank 1 bundle on M is trivial.

Exercise 3.5 (2 points). Let B be a vector bundle. Prove that $B \otimes B$ is oriented, or find a counterexample.

Exercise 3.6 (2 points). Construct a rank 2 vector bundle not admitting a non-degenerate bilinear form of signature (1,1), or prove that such bundle does not exist.

Exercise 3.7. Let $M_1 \stackrel{\phi}{\longrightarrow} M$ be a surjective, smooth map without critical points, M, M_1 compact manifolds, and B a non-trivial bundle on M. Can the pullback bundle $\phi^* B$ be trivial?

Exercise 3.8 (2 points). Construct a non-trivial rank 2 vector bundle which does not have any non-trivial sub-bundles.

4 Vector fields, diffeomorphisms and Frobenius form

Definition 4.1. Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0,a]$, and $V: M \times [0,a] \longrightarrow M$ a flow of diffeomorphisms which satisfies $(V_t^{-1})^* \frac{d}{dt} V_t = v_t$ for each $t \in [0,a]$, and $V_0 = \mathsf{Id}$. Then V_t is called **an exponent of** v_t.

Exercise 4.1 (2 points). Let V be a smooth vector field on $\mathbb{R}^2$, and e^{tV} it exponent. Prove that e^{tV} always exists for a sufficiently small interval $[0, \varepsilon[$ or find a counterexample.

Exercise 4.2. Let $B \subset TM$ be a 2-dimensional sub-bundle, and

$$\Phi: \Lambda^2 B \longrightarrow TM/B$$

its Frobenius form. Find $B \subset T\mathbb{R}^3$ such that Φ nowhere vanishes.

Exercise 4.3. Find a rank 1 sub-bundle $B \subset TS^3$ such that the corresponding foliation has no compact leaves.

Exercise 4.4. Find a rank 2 sub-bundle $B \subset TS^3$ such that the corresponding Frobenius form Φ nowhere vanishes.

5 Grassmann algebra

Definition 5.1. Let $\alpha \in \Lambda^2 V^*$ be a skew-symmetric 2-form on V. We say that α is **symplectic**, or **non-degenerate**, if for any $v \in V$, there exists $w \in V$ such that $\alpha(v, w) \neq 0$.

Exercise 5.1 (2 points). Let ω be a non-degenerate skew-symmetric 2-form on a $2n$-dimensional vector space V, and g a positive definite scalar product. Prove that there exists an orthonormal basis $x_1, ..., x_{2n}$ in V such that $\omega = \sum_{i=1}^n \alpha_i y_{2i-1} \wedge y_{2i}$, where y_i is the dual basis in V^*.

Exercise 5.2. Let $\eta \in \Lambda^k V$ be a non-zero form, and $L_\eta : \Lambda^1 \longrightarrow \Lambda^{1+k} V$ the multiplication map $x \longrightarrow x \wedge \eta$. Prove that its kernel $\ker L_\eta$ is at most k-dimensional.

Exercise 5.3. Let $\omega \in \Lambda^2 V^*$, $V = \mathbb{R}^{2n}$, $n \geqslant 2$ be a symplectic form, and $\alpha \in \Lambda^1 V^*$ a non-zero 1-form. Prove that $\alpha \wedge \omega \neq 0$.

Exercise 5.4. Let $V = \mathbb{R}^4$, and $\alpha \in \Lambda^2 V$, $\alpha \neq 0$. Prove that there exists $\beta \in \Lambda^2 V$ such that $\alpha \wedge \beta \neq 0$.

Exercise 5.5. Let $V = \mathbb{R}^4$, and $\alpha \in \Lambda^2 V^*$. Assume that $\alpha \wedge \alpha \in \Lambda^4 V^*$ is non-zero. Prove that α is a symplectic 2-form.

Exercise 5.6. Let $\omega_1, \omega_2 \in \Lambda^2(\mathbb{R}^4)$ symplectic forms satisfying $\omega_1 \wedge \omega_2 = 0$. Prove that $(\omega_1 + \omega_2) \wedge (\omega_2 - \omega_1) = 0$ or find a counterexample.