Differential geometry: home assignment 1

Rules: Home assignment.

Exercise 1.1. Prove that the Klein bottle cannot be smoothly embedded to $\mathbb{R}^3$.

Exercise 1.2 (*). Prove that the Klein bottle can be smoothly embedded to $\mathbb{R}^4$.

Exercise 1.3. Let $X, Y \subset \mathbb{R}^n$ be two closed subsets, $X \cap Y = \emptyset$. Find a continuous function which is equal to 1 on X and 0 on Y.

Exercise 1.4. Let $\phi: X \longrightarrow Y$ be a local diffeomorphism, with X and Y compact manifolds.

a. Prove that the set $\phi^{-1}(y)$ is finite for any $y \in Y$.

b. Prove that the number of elements in $\phi^{-1}(y)$ is independent from the choice of $y \in Y$.

Exercise 1.5. A map is called **open** if the image of any open set is open. Let $\phi: X \longrightarrow Y$ be a smooth map without critical points. Prove that ϕ is open.

Exercise 1.6. Prove that there exists a diffeomorhism from interior of a cube in $\mathbb{R}^n$ to interior of a ball.

Exercise 1.7. Prove that there is no surjective smooth map from a compact k-dimensional manifold to a compact n-dimensional manifold if $k < n$.