Differential geometry

lecture 2: partition of unity

Misha Verbitsky

Université Libre de Bruxelles

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Smooth manifolds in terms of maps and atlases (reminder)

DEFINITION: **Topological manifold** is a topological space which is locally homeomorphic to an open ball in $\mathbb{R}^n$.

DEFINITION: **An open cover** of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

DEFINITION: Let M be a topological manifold. A cover $\{U_i\}$ of M is an **atlas** if for every U_i, we have a map $\varphi_i : U_i \to \mathbb{R}^n$ giving a homeomorphism of U_i with an open subset in $\mathbb{R}^n$. In this case, one defines the **transition maps**

$$\Phi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

DEFINITION: A function $\mathbb{R} \longrightarrow \mathbb{R}$ is **of differentiability class C^i** if it is i times differentiable, and its i-th derivative is continuous. A map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **of differentiability class C^i** if all its coordinate components are. A **smooth function/map** is a function/map of class $C^\infty = \bigcap C^i$.

DEFINITION: An atlas is **smooth** if all transition maps are smooth (of class C^∞, i.e., infinitely differentiable), **smooth of class C^i** if all transition functions are of differentiability class C^i, and **real analytic** if all transition maps admit a Taylor expansion at each point.

Sheaves of functions (reminder)

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and presheaves: examples (reminder)

Examples of sheaves:

* Space of continuous functions

* Space of smooth functions, any differentiability class

* Space of real analytic functions

Examples of presheaves which are not sheaves:

* Space of constant functions **(why?)**

* Space of bounded functions **(why?)**

Ringed spaces (reminder)

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

CLAIM: Let (M, C^i) and (N, C^i) be manifolds of class C^i. Then **there is a bijection between smooth maps $f: M \longrightarrow N$ and the morphisms of corresponding ringed spaces.**

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ is a ring of functions on $\mathbb{R}^n$ of this class.

Embedded submanifolds (reminder)

DEFINITION: A **closed embedding** $\varphi: N \hookrightarrow M$ of topological spaces is an injective map from N to a closed subset $\varphi(N)$ inducing a homeomorphism of N and $\varphi(N)$.

DEFINITION: $N \subset M$ is called **a submanifold** of dimension m if for every point $x \in N$, there is a neighborhood $U \subset K$ diffeomorphic to an open ball, such that this diffeomorphism maps $U \cap N$ onto a linear subspace of dimension m.

REMARK: **Any submanifold $N \subset M$ is equipped with a structure of a manifold** induced from M.

DEFINITION: **A smooth embedding** $f: M \longrightarrow N$ of smooth manifolds is a closed embedding inducing a diffeomorphism of M to its image.

THEOREM: **(Whitney theorem)**
Any manifold can be smoothly embedded to $\mathbb{R}^n$.

Proven later today.

Locally finite covers

DEFINITION: An open cover $\{U_\alpha\}$ of a topological space M is called **locally finite** if every point in M possesses a neighborhood that intersects only a finite number of U_α.

Claim 1: Let $\{U_\alpha\}$ be an atlas on a manifold M. **Then there exists a refinement $\{W_\beta\}$ of $\{U_\alpha\}$ such that a closure of each W_β is compact in M.**

Proof: Let $\{U_\alpha\}$ be an atlas on M, and $U_\alpha \xrightarrow{\varphi_\alpha} \mathbb{R}^n$ homeomorphisms. Consider a cover $\{V_i\}$ of $\mathbb{R}^n$ given by open balls of radius 2 centered in integer points, and let $\{W_\beta\}$ be a cover of M obtained as union of $\varphi_\alpha^{-1}(V_i)$. ■

DEFINITION: Let $U \subset V$ be two open subsets of M such that the closure of U is contained in V. **In this case we write $U \Subset V$.**

DEFINITION: An open cover $\{U_\alpha\}$ of a topological space M is called **locally finite** if every point in M possesses a neighborhood that intersects only a finite number of U_α.

REMARK: If the atlas $\{U_\alpha\}$ considered in Claim 1 is locally finite then **the atlas $\{W_\beta\}$ is also locally finite.**

Locally finite covers and their refinements

THEOREM: Let $\{U_\alpha\}$ be a countable locally finite cover of a Hausdorff topological manifold, such that a closure of each U_α is compact, and each U_α is homeomorphic to $\mathbb{R}^n$. **Then there exists another cover $\{V_\alpha\}$ indexed by the same set such that $V_\alpha \Subset U_\alpha$.**

Proof. Step 1: Let $K_\alpha := M \backslash \bigcup_{\beta\neq\alpha} U_\beta$. By definition, K_α is closed. Sice $\{U_\alpha\}$ is a cover, $K_\alpha \subset U_\alpha$. Since the closure of U_α is compact, and $K_\alpha \subset U_\alpha$, the set K_α is compact. Therefore, K_α is contained in an open ball B_α of sufficiently big radius in $U_\alpha = \mathbb{R}^n$.

Step 2: Let $U_1, U_2, ...$ be all elements of the cover. Suppose that $V_1, ..., V_{n-1}$ is already found. To take an induction step it remains to find $V_n \Subset U_n$

Step 3: Replacing U_i by V_i and renumbering, we may assume that $n = 1$. **Then the statement of Theorem follows from Step 1** by taking $V_1 := B_1$, where B_1 is an open ball containing $K_1 := M \backslash \bigcup_{\beta\neq 1} U_\beta$. ■

Construction of a partition of unity

REMARK: If all U_α are diffeomorphic to $\mathbb{R}^n$, all V_α can be chosen diffeomorphic to an open ball. Indeed, any compact set is contained in an open ball.

COROLLARY: Let M be a manifold admitting a locally finite cover $\{U_\alpha\}$, with $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ diffeomorphisms. **Then there exists another atlas $\{U_\alpha, \varphi'_\alpha : U_\alpha \longrightarrow \mathbb{R}^n\}$, such that $\varphi'_\alpha(\mathbb{B})$ is also a cover of M, and $\mathbb{B} \subset \mathbb{R}^n$ a unit ball.** ■

EXERCISE: Find a smooth function $\nu : \mathbb{R}^n \longrightarrow [0,1]$ which vanishes outside of $\mathbb{B} \subset \mathbb{R}^n$ and is positive on $\mathbb{B}$.

REMARK: In assumptions of Corollary, let $\nu_\alpha(z) := \nu(\varphi'_\alpha)$, and $\mu_i := \frac{\nu_i}{\sum_\alpha \nu_\alpha}$. Then $\mu_\alpha : M \longrightarrow [0,1]$ are smooth functions with support in U_α satisfying $\sum_\alpha \mu_\alpha = 1$. Such a set of functions is called **a partition of unity**.

Partition of unity: a formal definition

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support satisfying the following conditions.
(a) Every function f_i has compact support in some U_i
(b) $\sum_i f_i = 1$

The argument of previous page proves the following theorem.

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

Whitney theorem for compact manifolds

THEOREM: Let M be a compact smooth manifold. **Then M admits a closed smooth embedding to $\mathbb{R}^N$.**

Proof. Step 1: Choose a finite atlas $\{V_i, \varphi_i : V_i \longrightarrow \mathbb{R}^n, i = 1, 2, ..., m\}$, and subordinate partution of unity $\mu_i : M \longrightarrow [0, 1]$.

Step 2: Denote by W_i the set $W_i := \{z \mid \mu_i(z) > \frac{1}{2m}\}$. **Since $\sum_{i=1}^m \mu_i = 1$, the set $\{W_i\}$ is a cover of M.** Let $\alpha : [0, 1] \longrightarrow [0, 1]$ be a smooth, monotonous function mapping 0 to 0 and $[1/2m, 1]$ to 1, and $\nu_i := \alpha(\mu_i)$. Then $\nu_i = 1$ on W_i.

Step 3: For each i, the map $\tilde{\Phi}_i(z) := \nu_i\varphi_i(z)$ **is smooth and induces a diffeomorphism of W_i and an open subset of $\mathbb{R}^n$.**

Step 4: The product map

$$\Psi := \prod_{i=1}^m : \Phi_i : M \longrightarrow \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times ... \times \mathbb{R}^n}_{m \text{ times}}$$

is an injective, continuous map from a compact, hence **it is a homeomorphism to its image.** It is a smooth embedding, because its differential is injective (use "implicit/inverse function theorem"). ∎

Embedding to $\mathbb{R}^\infty$

QUESTION: What if M is non-compact?

DEFINITION: Define $\mathbb{R}^I_f$ as a direct sum of several copies of $\mathbb{R}$ indexed by a set I, that is, the set of points in a product where only finitely meny of coordinates can be non-zero. **The set $\mathbb{R}^I_f$ has metric**

$$d((x_1, ..., x_n, ...), (y_1, ..., y_n, ...)) := \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + ... + |x_n - y_n| + ...}$$

It is well-defined, because only finitely many of x_i, y_i are non-zero.

THEOREM: Let M be a compact smooth manifold, $\{V_i, \varphi_i : V_i \longrightarrow \mathbb{R}^n, i \in I\}$ be a locally finite atlas, and $\mu_i : M \longrightarrow [0, 1]$ a subordinate partition of unity. Define $\nu_i := \alpha(\mu_i)$ and Φ_i as above, and let

$$\Psi := \prod_I : \Phi_i : M \longrightarrow \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times ... \times \mathbb{R}^n}_{I \text{ times}} \subset (\mathbb{R}^{n+1})^I$$

be the corresponding product map. Then **Ψ is a homeomorphism to its image.**

Embedding to $\mathbb{R}^\infty$ (cont.)

THEOREM: Let M be a compact smooth manifold, $\{V_i, \varphi_i : V_i \longrightarrow \mathbb{R}^n, i \in I\}$ be a locally finite atlas, and $\mu_i : M \longrightarrow [0,1]$ a subordinate partition of unity. Define $\nu_i := \alpha(\mu_i)$ and Φ_i as above, and let

$$\Psi := \prod_I : \Phi_i : M \longrightarrow \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times ... \times \mathbb{R}^n}_{I \text{ times}} \subset (\mathbb{R}^{n+1})^I$$

be the corresponding product map. Then **Ψ is a homeomorphism to its image.**

Proof. Step 1: Ψ is injective by construction. To prove that it is a homeomorphism, it suffices to check that an image of an open set U is open in $\Psi(M)$, for each $U \subset W_i$, for some open cover $\{W_i\}$

Step 2: However, the set $\Psi(W_i)$ is determined by $\nu_i(z) = 1$, that is, by $\Phi_i(z)_{n+1} = 1$, where $\Phi_i(z)_{n+1}$ is the last coordinate of $\Phi_i(z)$. Therefore, **Ψ maps W_i to an open subset of $\Psi(M)$.**

Step 3: Since $\Phi_i\big|_{\overline{W_i}}$ (restriction to a closure) is a continuous, bijective map from a compact, it's a homeomorphism. Therefore, **an image of any open subset $U \subset W_i$ is open in $\Psi(W_i)$, which is open in $\Psi(M)$ as follows from Step 2.** ■

Measure 0 subsets and Sard's theorem

DEFINITION: A subset $Z \subset \mathbb{R}^n$ has **measure zero** if, for every $\varepsilon > 0$, there exists a countable cover of Z by open balls U_i such that $\sum_i \operatorname{Vol} U_i < \varepsilon$.

DEFINITION: A subset $Z \subset M$ of a manifold M has **measure 0** if intersection of M with each chart $U_i \hookrightarrow \mathbb{R}^n$ has measure 0.

Properties of measure 0 subsets.

A countable union of measure 0 subsets has measure 0.

A measure 0 subset $Z \subset M$ satisfies $(M \backslash Z) \cap U \neq \emptyset$ for any non-empty open subset $U \subset M$.

THEOREM: (a special case of Sard's Lemma) Let $f: M \longrightarrow N$ be a smooth map of manifolds, $\dim M < \dim N$. **Then $f(M)$ has measure zero in N.**

EXERCISE: Prove it.

Whitney's theorem (with a bound on dimension): strategy of the proof

THEOREM: Let M be a smooth n-manifold. **Then M admits a closed embedding to $\mathbb{R}^{2n+2}$.**

Strategy of the proof:

1. M is embedded to $\mathbb{R}^\infty$.
2. We find a linear projection $\mathbb{R}^\infty \xrightarrow{\pi} \mathbb{R}^{2n+2}$ such that $\pi|_M$ is a closed embedding of manifolds.

LEMMA: Let $M \subset \mathbb{R}^I$ be a subset, and $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^J$ a linear projection. Consider the set W of all vectors $\mathbb{R}(x-y)$, where $x, y \in M$ are distinct points. **Then $\pi|_M$ is injective if and only if $\ker \pi \cap W = 0$.**

Proof:: $\pi|_M$ is not injective if and only if $\pi(x) = \pi(y)$, which is equivalent to $\pi(x-y) = 0$. ■

Whitney's theorem: injectivity of projections

REMARK: Let $M \subset \mathbb{R}^I$ be a submanifold, and $W \subset \mathbb{R}^I$ the set of all vectors $\mathbb{R}(x-y)$, where $x, y \in M$ are distinct points. **Then W is an image of a $2m+1$-dimensional manifold**, hence (by Sard's Lemma) **for any projection of $\mathbb{R}^I$ to a $(2m+2)$-dimensional space, image of W has measure 0.**

COROLLARY: Let $M \subset \mathbb{R}^I$ be an m-dimensional submanifold, and $S \subset \mathbb{R}^I$ a maximal linear subspace not intersecting W. **Then the projection of W to $\mathbb{R}^I/S$ is surjective.**

Proof:: Suppose it's not surjective: $v \notin S$. Then $S \oplus \mathbb{R}v$ satisfies assumptions of lemma, hence $M \longrightarrow \mathbb{R}^I/(S + \mathbb{R}v)$ is also injective. ■

THEOREM: Let M be a smooth n-manifold, $M \hookrightarrow \mathbb{R}^I$ an embedding constructed earlier. **Then there exists a projection $\pi : \mathbb{R}^I \longrightarrow \mathbb{R}^{2n+2}$ which is injective on M.**

Proof:: Let S be the maximal linear subspace such that the restriction of $\pi : \mathbb{R}^I \longrightarrow \mathbb{R}^I/S$ to M is injective. Then the $2m+1$-dimensional manifold W surjects to $\mathbb{R}^I/S$, hence $\dim \mathbb{R}^i/S \leqslant 2m+1$ by Sard's lemma. ■

Tangent space to an embedded manifold

DEFINITION: Let $M \hookrightarrow \mathbb{R}^n$ be a smooth m-submanifold. The **tangent plane** at $p \in M$ is the plane in $\mathbb{R}^n$ tangent to M (i.e, the plane lying in the image of the differential given in local coordinates). A **tangent vector** is an arbitrary vector in this plane with the origin at p. The space of all tangent vectors at p is denoted by T_pM. Given a metric on $\mathbb{R}^n$, we can define the space of **unit tangent vectors** $\mathbb{S}^{m-1}M$ as the set of all pairs (p,v), where $p \in M$, $v \in T_pM$, and $|v| = 1$.

REMARK: $\mathbb{S}^{m-1}M$ is a smooth manifold, projected to M with fibers isomorphic to $m-1$-spheres, hence **$\mathbb{S}^{m-1}M$ is $(2m-1)$-dimensional.**

LEMMA: Let $M \subset \mathbb{R}^I$ be a subset, and $\pi : \mathbb{R}^I \longrightarrow \mathbb{R}^J$ a linear projection. Consider the set W' of all vectors $\mathbb{R}t$, where $t \in T_xM$ **Then the differential $D\pi|_M$ is injective if and only if $\ker \pi \cap W' = 0$.** ■

Now the above argument is repeated: we take a maximal space $S \supset \mathbb{R}^I$ such that the restriction of $\pi : \mathbb{R}^I \longrightarrow \mathbb{R}^I/S$ to M is injective and has injective differential, and the projection of $W \cup W'$ to $\mathbb{R}^I/S$ has to be surjective. However, W' is an image of an $2m$-dimensional manifold $\mathbb{S}^{m-1}M \times \mathbb{R}$, hence **the projection of $W \cup W'$ to $\mathbb{R}^I/S$ can be surjective only if $\dim \mathbb{R}^I/S \leqslant 2m+2$.**

This proves Whitney's theorem.