Differential geometry

lecture 3: vector fields

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Rings and derivations

REMARK: All rings in these lectures are assumed to be commutative and with unit. Algebras are associative, but not necessarily commutative (such as the matrix algebra). **Rings over a field k** are rings containing a field k. We assume that k has characteristic 0.

DEFINITION: Let A be a ring over a field k. A k-linear map $D\ A \longrightarrow A$ is called **a derivation** if it satisfies **the Leibniz identity** $D(fg) = D(f)g + gD(f)$. The space of derivations is denoted as $\operatorname{Der}_k(A)$.

EXAMPLE: $\frac{d}{dt}: \mathbb{C}[t] \longrightarrow \mathbb{C}[t]$. $\frac{d}{dt}: C^\infty\mathbb{R} \longrightarrow C^\infty\mathbb{R}$.

REMARK: Any derivation $\delta \in \operatorname{Der}_k(A)$ vanishes on $k \subset A$. Indeed, $\delta(1) = \delta(1 \cdot 1) = 2\delta(1)$.

CLAIM: Let K be **a finite extension** of a field k, that is, a field containing k and finite-dimensional as a K-linear space. **Then $\operatorname{Der}_k(K) = 0$.**

Proof: Indeed, any $x \in K$ satisfies a non-trivial polynomial equation $P(x) = 0$ with coefficients in k. Chose $P(t)$ of smallest degree possible. **For any $\delta \in \operatorname{Der}_k(A)$, we have $0 = \delta(P(x)) = P'(x)\delta(x)$,** and unless $\delta(x) = 0$, one has $P'(x) = 0$, giving a contradiction. ■

Modules over a ring

DEFINITION: Let A be a ring over a field k. **An A-module** is a vector space V over k, equipped with an algebra homomorphism $A \longrightarrow \operatorname{End}(V)$, where $\operatorname{End}(V)$ denotes the endomorphism algebra of V, that is, the matrix algebra.

REMARK: Let A be a field. Then A-modules are the same as vector spaces over A.

DEFINITION: Homomorphisms, isomorphisms, submodules, quotient modules, direct sums of modules are defined in the same way as for the vector spaces. A ring A is itself an A-module. A direct sum of n copies of A is denoted A^n. Such A-module is called **a free A-module**.

EXAMPLE: A-submodules in A are the same as ideals in A.

DEFINITION: Finitely generated A-module is a quotient module of A^n.

REMARK: One could generalize the definition of derivations by considering **derivatiobs with values in a module** as maps $\delta : A \longrightarrow V$ satisfying the Leibnitz identity.

Derivations as an A-module

REMARK: Let A be a ring over k. **The space $\operatorname{Der}_k(A)$ of derivations is also an A-module,** with multiplicative action of A given by $rD(f) = rD(f)$.

CLAIM: Let $A = k[t_1, .., t_k]$ be a polynomial ring. **Then $\operatorname{Der}_k(A)$ is a free A-module isomorphic to A^n,** with generators $\frac{d}{dt_1}, \frac{d}{dt_2}, ..., \frac{d}{dt_n}$.

Proof: Consider a map $\operatorname{Der}_k(A) \longrightarrow A^n$,

$$D \longrightarrow (D(t_1), D(t_2), ..., D(t_n))$$

It is surjective, because it maps each $\frac{d}{dt_i}$ to $(0, ..., 0, 1, 0, ..., 0)$, and injective, because each derivation which vanishes on t_i, vanishes on the whole polynomial ring. ∎

Now we prove a similar result for $C^\infty \mathbb{R}^n$.

Hadamard's Lemma

LEMMA: (Hadamard's Lemma)
Let f be a smooth function on $\mathbb{R}^n$, and x_i the coordinate functions. **Then $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, for some smooth $g_i \in C^\infty \mathbb{R}^n$.**

Proof: Let $t \in \mathbb{R}^n$. Consider a function $h(t) \in C^\infty \mathbb{R}^n$, $h(t) = f(tx)$. Using the chain rule, we get $\frac{dh}{dt} = \sum \frac{d}{dx_i} f(tx) x_i$, obtaining

$$f(x) - f(0) = \int_0^1 \frac{dh}{dt} dt = \sum_i x_i \int_0^1 \frac{df(tx)}{dx_i}(tx) dt.$$

■

COROLLARY: Let $\mathfrak{m}_0$ be an ideal of all smooth functions on $\mathbb{R}^n$ vanishing in 0. **Then $\mathfrak{m}_0$ is generated by coordinate functions.** ■

COROLLARY: Let f be a smooth function on $\mathbb{R}^n$ satisfying $f(x) = 0$ and $f'(x) = 0$. **Then $f \in \mathfrak{m}_x^2$.**

Proof: $f(x) = \sum_{i=1}^n x_i g_i(x)$, where all g_i vanish in 0. ■

Derivations of $C^\infty\mathbb{R}^n$

THEOREM: Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$, $A = C^\infty\mathbb{R}^n$, and $\operatorname{Der}(A) \xrightarrow{\Psi} (C^\infty\mathbb{R}^n)^n$ map D to $(D(x_1), D(x_2), ..., D(x_n))$. **Then $\Psi: \operatorname{Der}(C^\infty\mathbb{R}^n) \longrightarrow A^n$ is an isomorphism.**

Proof. Step 1: Since Ψ maps each $\frac{d}{dt_i}$ to $(0, ..., 0, 1, 0, ..., 0)$, it is **surjective.**

Step 2: Let $\mathfrak{m}_0$ be an ideal of functions vanishing in 0, and $\delta \in \ker\Psi$. Then $\delta(x_i) = 0$. By Hadamard's Lemma, $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, hence $\delta(f) = \sum_{i=1}^n x_i \delta(g_i)$. **Therefore, $\delta(f)$ lies in $\mathfrak{m}_0$.**

Step 3: Same argument proves that $\delta(f)$ vanishes everywhere, for all $f \in C^\infty M$. ■

Vector fields

DEFINITION: Let M be a smooth manifold. A **vector field** on M is an element in Der$(C^\infty M)$.

EXAMPLE: For $M = \mathbb{R}^n$, **the space** Der$(C^\infty M)$ **is a free module generated by** $\frac{d}{dx_i}$**,** $i = 1, ..., n$**.**

DEFINITION: A function f is called **supported in** $K \subset M$ if each point $x \notin K$ has a neighbourhood U such that 0 on U.

DEFINITION: Vector fields with support $K \subset M$ are vector fields $X \in$ Der$(C^\infty M)$ such $X(C^\infty M)$ takes values in functions supported in K.

REMARK: Vector fields with compact support K which lies in an open set $U \subset M$ diffeomorphic to $\mathbb{R}^n$ **are identified with** $\sum \alpha_i \frac{d}{dx_i}$**,** where α_i are functions with compact support in K, and x_i coordinates on $U = \mathbb{R}^n$.

REMARK: To express this, one says that "vector fields are locally free over functions", or **"vector fields are locally free sheaf of** $C^\infty M$**-modules."** In the next lecture I will explain what it means.

Tangent space

DEFINITION: Define **the tangent space** T_xM to a manifold M in a point $x \in M$ as the space of all derivations $\operatorname{Der}(C^\infty M, C^\infty M/\mathfrak{m}_x)$ mapping $C^\infty M$ to $\mathbb{R} = C^\infty M/\mathfrak{m}_x$, where $\mathfrak{m}_x$ denotes the ideal of all functions vanishing in x.

REMARK: Clearly, **any derivation $C^\infty M \longrightarrow \mathbb{R} = C^\infty M/\mathfrak{m}_x$ vanishes on the square $\mathfrak{m}_x^2$ of the ideal $\mathfrak{m}_x$.**

REMARK: From Hadamard's lemma it follows that $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim M$, and, moreover, the space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is generated by coordinate functions **(check this)**.

CLAIM: For any $x \in M$, **one has $T_xM = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.**

Proof: Any $C^\infty M/\mathfrak{m}_x$-valued derivation vanishing on $\mathfrak{m}_x^2$ gives a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \mathbb{R}$. Moreover, any such map satisfies Leibnitz identity **(check this)**. Therefore, $\operatorname{Der}(C^\infty M, C^\infty M/\mathfrak{m}_x) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

REMARK: Any vector field defines a derivation $\operatorname{Der}(C^\infty M, C^\infty M/\mathfrak{m}_x)$ in each point $x \in M$. Moreover, vector fields are identified with the families of such derivations smoothly depending on $x \in M$.

Locally trivial fibrations

DEFINITION: A smooth map $f: X \longrightarrow Y$ is called **a locally trivial fibration** if each point $y \in Y$ has a neighbourhood $U \ni y$ such that $f^{-1}(U)$ is diffeomorphic to $U \times F$, and the map $f: f^{-1}(U) = U \times F \longrightarrow U$ is a projection. In such situation, F is called **the fiber** of a locally trivial fibration.

DEFINITION: A trivial fibration is a map $X \times Y \longrightarrow Y$.

EXAMPLE: The projection $S^3 \subset \mathbb{C}^2 \backslash 0 \xrightarrow{f} \mathbb{C}P^1$ is called **the Hopf fibration**. Given $U = \{x : 1\} \subset \mathbb{C}P^1$, with $|x| \leqslant 1$, one has

$$f^{-1}(U) = \{z_1, z_2 \in S^3 \ \ | \ \ |z_1|^2 + |z_2|^2 = 1, |z_1| \leqslant 1\}$$

(here z_i are complex coordinates in $\mathbb{C}^2$). Then

$$f^{-1}(U) = \left\{(z_1, z_2) \ \ | \ \ z_2 \in U(1) \cdot \sqrt{1 - |z_1|^2}\right\},$$

where $U(1) = \{z \in \mathbb{C} \ \ | \ \ |z| = 1\}$. Therefore, **the Hopf fibration $f: S^3 \longrightarrow S^2$ is a locally trivial fibration**.

REMARK: Since $\pi_1(S^3) = 0$ and $\pi_1(S^1 \times S^2) = \mathbb{Z}$, **the Hopf fibration is non-trivial.**

Vector bundles

DEFINITION: A vector bundle on Y is a locally trivial fibration $f: X \longrightarrow Y$ with fiber $\mathbb{R}^n$, with each fiber $V := f^{-1}(y)$ equipped with a structure of a vector space, smoothly depending on $y \in Y$.

EXAMPLE: Let TM be the set of all tangent vectors $v \in T_x M$. We equip TM with a topology in such a way that locally it is a product of $U \subset M$ and $\mathbb{R}^n$, with basis given by a set of pointwise linearly independent vector fields (locally such a basis always exists). Then TM is called **the tangent bundle** to M.

REMARK: In this setup, **vector fields are identified with the set of smooth sections of the standard projection $\pi: TM \longrightarrow M$.**

Tangent bundle: examples.

EXAMPLE: $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Indeed, if coordinate vector fields d/dx_i form a basis in $T\mathbb{R}^n$. However, **if TM admits a basis globally in M, the bundle is trivial, because it can be expressed as a product.**

EXAMPLE: Let $\mathbb{R}^n/\mathbb{Z}^n$ be a torus. Since the coordinate vector fields are invariant under the $\mathbb{Z}^n$-action on $\mathbb{R}^n$, they give a basis in TM everywhere on $\mathbb{R}^n/\mathbb{Z}^n$. **Therefore, $T\mathbb{R}^n/\mathbb{Z}^n = \mathbb{R}^n \times (T\mathbb{R}^n/\mathbb{Z}^n)$.**

EXAMPLE: TS^2 is non-trivial; otherwise, one could "comb the hedgehog" (find a nowhere vanishing vector field on a 2-sphere).