Differential geometry

lecture 4: sheaves

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Sheaves

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below **is a more abstract version of the notion of "sheaf of functions"** defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Ringed spaces (reminder)

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A sheaf of functions is a subsheaf in the sheaf of all functions, closed under multiplication.

For simplicity, I assume that the sheaves of rings we consider are all subsheaves in the sheaf of all functions.

DEFINITION: A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M, \mathcal{F}) \stackrel{\Psi}{\longrightarrow} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \stackrel{\Psi}{\longrightarrow} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Smooth manifolds (reminder)

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: Diffeomorphism of smooth manifolds is a homeomorphism φ which induces an isomorphisms of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

We assume by default that all manifolds are Hausdorff and of class C^∞.

Partition of unity (reminder)

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support satisfying the following conditions.
(a) Every function f_i has compact support in some U_i
(b) $\sum_i f_i = 1$

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

DEFINITION: Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

DEFINITION: Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$ on M. A point $x \in M$ does not lie in the **support** $\operatorname{Sup}(f)$ of f if $f|_U = 0$ for some neighbourhood $U \ni x$. A section is called **section with compact support** or **supported on a compact set** if its support is compact.

REMARK: **Support of a section is obviously closed.**

Vector fields as derivations

DEFINITION: Let M be a smooth manifold. A **vector field** on M is an element in $\operatorname{Der}(C^\infty M)$.

EXAMPLE: For $M = \mathbb{R}^n$, **the space $\operatorname{Der}(C^\infty M)$ is a free module generated by $\frac{d}{dx_i}$, $i = 1, ..., n$.**

Pros of this definition: it is entirely coordinate-free.
Cons: Restriction to an open subset is a complicated business.

THEOREM: Let $U \Subset V$ be open subset of a smooth metrizable manifold, and $D \in \operatorname{Der}(C^\infty M)$ a derivation. Consider a smooth function $\Phi_{U,V} \in C^\infty M$ supported on V, and equal to 1 on U. Given $f \in C^\infty V$, define $D(f)|_U := D(\Phi_{U,V} f)$. Choosing a cover $\{U_i\}$ of such sets, we can glue together a section $D(f)$ of $C^\infty V$ from such $D(f)\Big|_{U_i}$. **This operation is independent of all the choices we made and gives an element $D|_V \in \operatorname{Der}(V)$. Moreover, such restriction maps define a structure of a sheaf on $\operatorname{Der}(M)$.**

Proof: Next lecture.

Morphisms of sheaves

DEFINITION: Let $\mathcal{B}, \mathcal{B}'$ be sheaves on M. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

DEFINITION: **A sheaf isomorphism** is a homomorphism $\Psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi: \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$.

Sheaves of modules

REMARK: Let $A: \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \stackrel{\varphi_{U,U'}}{\longrightarrow} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: **A vector bundle** on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Smooth functions with prescribed support

EXERCISE: Let $X, Y \subset M$ be non-intersecting closed subsets in a metric space. **Find non-intersecting open neighbourhoods $U_1 \supset X$ and $U_2 \supset Y$.**

Proposition 1: Let $U \Subset V$ – open subsets in a smooth metrizable manifold. **Then there exists a smooth function $\Phi_{U,V} \in C^\infty M$, supported on V, and equal to 1 on U.**

Proof. Step 1: Let U_1, U_2 be non-intersecting open sets containing the closure $X = \overline{U}$ and $Y = M \backslash V$, and $U_3 = V \backslash \overline{U}$. Since $U_1 \cup U_2$ contains $\overline{U}$ and $M \backslash V$, **U_1, U_2, U_3 is a cover of M.**

Step 2: Consider a cover of M by open sets $\{V_i\}$ which are contained in either U_1, U_2 or U_3, but never intersect both U_1 and U_2. Let ψ_i be a partition of unity supported in V_i, and $\Phi_{U,V}$ be the sum of all ψ_i with support in $U_1 \cap U_3$ and intersecting U_2. Then $\Phi_{U,V} = 0$ in $M \backslash U_2 \supset M \backslash V$, because support of $\Phi_{U,V}$ does not intersect U_2, and $\Phi_{U,V} = 1$ on $U_1 \supset U$, because $\Phi_{U,V}$ is a sum of all ψ_i with support intersecting U_2. ■

Local operators

DEFINITION: A linear map $\Psi: C^\infty(M) \longrightarrow C^\infty(M)$ is called **local** if for any function f supported in a compact subset $Z \subset M$, its image $\Psi(f)$ is supported in Z.

LEMMA: Any derivation $D: C^\infty(M) \longrightarrow C^\infty(M)$ is local.

Proof: Let f be a function supported in Z. For each g with support outside of Z, we have $0 = D(fg) = fD(g) + gD(f)$.

Proposition 1 gives a function g which is equal to 1 on any open subset U with its closure not intersecting Z, and 0 in a neighbourhood of Z. Then $fg = 0$. This gives $0 = D(fg)|_U = gD(f)|_U = D(f)|_U$. **Therefore, U does not intersect support $D(f)$.** However, the union of all such U is $M \backslash Z$. ■

REMARK: By definition, **differential operator** is an operator expressed through derivations and multiplication by a function. From the above lemma, we obtain that **all differential operators are local**. The converse is also true, but harder to prove: **any local operator on $C^\infty M$ is a differential operator**.