Differential geometry

lecture 5: tangent bundle as a sheaf

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November 7, 2016

Sheaves (reminder)

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below **is a more abstract version of the notion of "sheaf of functions"** defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Ringed spaces (reminder)

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A sheaf of functions is a subsheaf in the sheaf of all functions, closed under multiplication.

For simplicity, I assume that the sheaves of rings we consider are all subsheaves in the sheaf of all functions.

DEFINITION: A **ringed space** $(M,\mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M,\mathcal{F}) \stackrel{\Psi}{\longrightarrow} (N,\mathcal{F}')$ of ringed spaces is a continuous map $M \stackrel{\Psi}{\longrightarrow} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Smooth manifolds (reminder)

DEFINITION: Let $(M,\mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M,\mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n,\mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: **Diffeomorphism** of smooth manifolds is a homeomorphism φ which induces an isomorphisms of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

We assume by default that all manifolds are Hausdorff and of class C^∞.

Morphisms of sheaves (reminder)

DEFINITION: Let $\mathcal{B}, \mathcal{B}'$ be sheaves on M. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \downarrow & & \downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

DEFINITION: **A sheaf isomorphism** is a homomorphism $\Psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi : \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$.

Sheaves of modules (reminder)

REMARK: Let $A : \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\varphi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: **A vector bundle** on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Smooth functions with prescribed support (reminder)

EXERCISE: Let $X, Y \subset M$ be non-intersecting closed subsets in a metric space. **Find non-intersecting open neighbourhoods $U_1 \supset X$ and $U_2 \supset Y$.**

Proposition 1: Let $U \Subset V$ – open subsets in a smooth metrizable manifold. **Then there exists a smooth function $\Phi_{U,V} \in C^\infty M$, supported on V, and equal to 1 on U.**

Proof. Step 1: Let U_1, U_2 be non-intersecting open sets containing the closure $X = \overline{U}$ and $Y = M \backslash V$, and $U_3 = V \backslash \overline{U}$. Since $U_1 \cup U_2$ contains $\overline{U}$ and $M \backslash V$, **U_1, U_2, U_3 is a cover of M.**

Step 2: Consider a cover of M by open sets $\{V_i\}$ which are contained in either U_1, U_2 or U_3, but never intersect both U_1 and U_2. Let ψ_i be a partition of unity supported in V_i, and $\Phi_{U,V}$ be the sum of all ψ_i with support in $U_1 \cap U_3$ and intersecting U_2. Then $\Phi_{U,V} = 0$ in $M \backslash U_2 \supset M \backslash V$, because support of $\Phi_{U,V}$ does not intersect U_2, and $\Phi_{U,V} = 1$ on $U_1 \supset U$, because $\Phi_{U,V}$ is a sum of all ψ_i with support intersecting U_2. ■

Local operators

DEFINITION: A linear map $\Psi: C^\infty(M) \longrightarrow C^\infty(M)$ is called **local** if for any function f supported in a compact subset $Z \subset M$, its image $\Psi(f)$ is supported in Z.

LEMMA: Any derivation $D: C^\infty(M) \longrightarrow C^\infty(M)$ is local.

Proof: Let f be a function supported in Z. For each g with support outside of Z, we have $0 = D(fg) = fD(g) + gD(f)$.

Proposition 1 gives a function g which is equal to 1 on an open subset U with its closure not intersecting Z, and 0 in a neighbourhood of Z. Then $fg = 0$. This gives $0 = D(fg)|_U = gD(f)|_U$ (because $fD(g)|_U = 0$), and $0 = gD(f)|_U = D(f)|_U$ because $g|_U = 1$.

Therefore, U does not intersect support $D(f)$. However, the union of all such U is $M \backslash Z$. ∎

Vector fields as derivations

DEFINITION: Let M be a smooth manifold. A **vector field** on M is an element in $\operatorname{Der}(C^\infty M)$.

EXAMPLE: For $M = \mathbb{R}^n$, **the space** $\operatorname{Der}(C^\infty M)$ **is a free module generated by** $\frac{d}{dx_i}$**,** $i = 1, ..., n$**.**

REMARK: By definition, **differential operator** is an operator expressed through derivations and multiplication by a function. From the above lemma, we obtain that **all differential operators are local**. The converse is also true, but harder to prove: **any local operator on** $C^\infty M$ **is a differential operator**.

Sections with compact support

CLAIM: Let $U \subset V$ be open subsets of a Hausdorff space M. **A section $f \in \mathcal{F}(U)$ with compact support $Z \subset U$ can be uniquely extended to $\tilde{f} \in \mathcal{F}(V)$, also with support in Z.**

Proof: Consider a cover $\{U_1 = U, U_2 = V \backslash Z\}$ of V, and let $f_1 = f \in F(U_1)$ and $f_2 = 0 \in F(U_2)$. Since $f_i\Big|_{U_1 \cap U_2} = 0$, we can glue f_1 and f_2, obtaining the extension $\tilde{f}$ and $0 \in \mathcal{F}(U_2)$. ■

DEFINITION: Let F, G be sheaves of $C^\infty M$-modules. For any open $U \subset M$ denote by $\operatorname{Hom}_c(F, G)|_U$ the space of $\mathbb{R}$ or $\mathbb{C}$-linear maps from the space of sections of F over U with compact support to $G|_U$. For any $V \subset U$, we define **the restriction map** $\operatorname{Hom}_c(F, G)|_U \longrightarrow \operatorname{Hom}_c(F, G)|_V$ by taking any section of F with compact support on V, extending it to U, applying $\varphi \in \operatorname{Hom}_c(F, G)|_U$ and restricting the resulting section of G to V.

Sections with compact support

CLAIM: $U \mapsto \operatorname{Hom}_c(F,G)|_U$ **defines a sheaf of C^∞-modules.**

Proof. Step 1: Suppose that $\varphi \in \operatorname{Hom}_c(F,G)|_U$ vanishes locally on a covering $\{U_i\}$ of U, and $f \in F|_U$ a section with compact support. Choosing a subordinate partition of unity, we express f as a sum $\sum \zeta_i f$, where ζ_i have support in U_i (this sum can be chosen finite because f has compact support). Then $\varphi(f) = \sum \varphi(\zeta_i f) = 0$. **This takes care of the first sheaf axiom.**

Step 2: Suppose $\{U_i\}$ is a covering of U, where $\varphi_i \in \operatorname{Hom}_c(F,G)\Big|_{U_i}$ are maps compatible over $U_i \cap U_j$. For any $f \in F|_U$ with compact support, we define $\varphi(f) := \sum \varphi(\zeta_i f)$, where ζ_i is the subordinate partition of unity. **This gives gluing of sections,** proving the second sheaf axiom. ∎

Locality of derivations and partition of unity

Claim 1: Let $U \subset V$ be an open set, and $\psi \in C^\infty V$ a function with compact support such that $\psi|_U = 1$. Then **for any derivation $D \in \operatorname{Der}(C^\infty V)$ and any $f \in C^\infty V$, the restriction $D(\psi f)|_U$ is independent from the choice of ψ.**

Proof: Let ψ_1, ψ_2 be two such functions, then $\operatorname{Sup}(\psi_1 f - \psi_2 f) \cap U = \emptyset$, and by locality we obtain $\operatorname{Sup}(D(\psi_1 f) - D(\psi_2 f)) = \emptyset$. ■

COROLLARY: Let V be an open subset of a smooth manifold M, and $D \in \operatorname{Der}(C^\infty V)$. Consider a partition of unity ψ_i on V. **Then $D(f) = \sum D(\psi_i f)$.** ■

REMARK: This statement is not entirely trivial, because the sum can be infinite, but **it makes sense because D is local**. Indeed, for any open set U which intersects only finitely many $\operatorname{Sup} \psi_i$, say, $\psi_1, ..., \psi_n$, one has $f|_U = \sum_{i=1}^n \psi_i f$. By Claim 1, this gives $D(f)|_U = \sum_{i=1}^n D(\psi_i f)|_U$. Then $D(f) = \sum D(\psi_i f)$ everywhere on M.

Restriction of a derivation

CLAIM: Let D be a derivation of $C^\infty M$. Consider D as a section of the sheaf $\operatorname{Hom}_c(C^\infty M, C^\infty M)$ defined above. **Then the restriction of D to $U \subset M$ gives a derivation on $C^\infty U$.**

Proof. Step 1: Fix a partition of unity ψ_i on U, and let $f \in C^\infty U$ Since $\sum \psi_i = 1$, **the sum $\sum D(\psi_i f)$ is independent from the choice of a partition.** Indeed, for any open set intersecting only finitely many of $\operatorname{Sup} \psi_i$, the sum $\sum D(\psi_i f)|_U$ is finite. It is independent from the choice of ψ_i by Claim 1 above.

Proof. Step 2: $D(fg) = \sum D(\psi_i fg) = \sum D(\psi_i f)g + \sum_j \psi_j f \sum_i D(\psi_i g) = \sum f D(\psi_i g) + \sum g D(\psi_i f)$, hence it is a derivation. ■

Tangent bundle

COROLLARY: The sheaf of derivations is locally free, that is, $\operatorname{Der} C^\infty M$ **defines a vector bundle on** M**.**

Proof: Indeed, the module of derivations is free on $\mathbb{R}^n$. ■

DEFINITION: It is called **the tangent bundle**, and denoted TM.

REMARK: The notions of "vector bundle" in lecture 3 and lecture 4 are different, but equivalent. We shall reconcile the difference in the next lecture.