Differential geometry

lecture 7: flows of diffeomorphisms and Frobenius theorem

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Flow of diffeomorphisms

DEFINITION: Let $f: M \times [a,b] \longrightarrow M$ be a smooth map such that for all $t \in [a,b]$ the restriction $f_t := f\big|_{M\times\{t\}}: M \longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f \in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c}: C^\infty M \longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^* \frac{dV_t}{dt}|_{t=c} f$. **Then $f \longrightarrow (V_t^{-1})^* \frac{d}{dt} V_t^* f$ is a derivation** (that is, a vector field).

Proof: $\frac{d}{dt}V_t^*(fg) = V_t^*(f)\frac{d}{dt}V_t^*g + \frac{d}{dt}V_t^* f V_t^*(g)$ by the Leibnitz rule, giving

$$(V_t^{-1})^* \frac{d}{dt} V_t^*(fg) = f(V_t^{-1})^* \frac{d}{dt} V_t^* g + g(V_t^{-1})^* \frac{d}{dt} V_t^* f.$$

■

DEFINITION: The vector field $\frac{d}{dt}V_t|_{t=c}$ is called **a vector field tangent to a flow of diffeomorphisms V_t at $t = c$.**

DEFINITION: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [a,b]$, and $V: M \times [a,b] \longrightarrow M$ a flow of diffeomorphisms which satisfies $(V_t^{-1})^* \frac{d}{dt} V_t = v_t$ for each $t \in [a,b]$, and $V_0 = \mathrm{Id}$. Then V_t is called **an exponent of v_t**.

Automorphisms of the ring of functions

REMARK: Each diffeomorphism $\psi: M \longrightarrow M$ induces an automorphism of the ring of smooth functions on M, $f \mapsto \psi^* f$.

THEOREM: Let M be a compact manifold. Then **any automorphism $\Psi: C^\infty M \longrightarrow C^\infty M$ is induced by a diffeomorphism of M.**

Proof. Step 1: Given a point $x \in M$, denote by I_x **the maximal ideal of x**, that is, the ideal of all functions vanishing in x. On a compact manifold, any maximal ideal is obtained this way. Indeed, if an ideal $I \subset C^\infty M$ has no common zeros, for each $y \in M$ there exists $f_y \in I$ which does not vanish in y. Denote by U_y the open set where $f_y \neq 0$. Then $\{U_y\}$ is an open cover of M. Finding a finite subcover, we obtain a finite number of functions $f_i \in I$ such that $\bigcap_i U_{f_i} = M$. Then **the function $\sum f_i^2 \in I$ is invertible, hence $I = C^\infty M$ is not a maximal ideal.**

Step 2: Identifying points and maximal ideals, we obtain a map $\psi: M \longrightarrow M$ induced by Ψ. **It remains to show that this map is a diffeomorphism.**

Automorphisms of the ring of functions (2)

THEOREM: Let M be a compact manifold. Then **any automorphism $\Psi: C^\infty M \longrightarrow C^\infty M$ is induced by a diffeomorphism of M.**

Step 2: Identifying points and maximal ideals, we obtain a map $\psi: M \longrightarrow M$ induced by Ψ. **It remains to show that this map is a diffeomorphism.**

Step 3: All open subsets of M can be obtained as unions of open sets $U_f := f^{-1}(\mathbb{R}\backslash 0)$, where $f \in C^\infty M$ **(prove it)**. Fowever, $f(x) = 0$ if and only if $f \in I_x$. Then U_f can be considered as a set of maximal ideals I_x such that $f \notin I_x$. Since Ψ maps U_f to $U_{\Psi(f)}$, the corresponding map ψ is continuous on M. This implies that ψ **is a homeomorphism.**

Step 4: Finally, Ψ maps coordinate functions on $U \subset M$ to coordinate functions on $\psi^{-1}(U)$, hence this homeomorphism is smooth. ■

Solutions of ODE (1)

DEFINITION: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0, a]$, and $V: M \times [0, a] \longrightarrow M$ a flow of diffeomorphisms which satisfies $(V_t^{-1})^* \frac{d}{dt} V_t = v_t$ for each $t \in [0, a]$, and $V_0 = \mathrm{Id}$. Then V_t is called **an exponent of v_t**.

REMARK: To take an exponent of a vector field **is the same as to solve an ordinary differential equation.**

Theorem 1: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0, a]$. Then **the exponent of v_t is unique. It always exists** when v_t has compact support.

Solutions of ODE (2)

Theorem 1: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0, a]$. Then **the exponent of v_t is unique. It always exists** when v_t vanish (for all t) outside of a compact set $K \subset M$.

Proof: To construct a flow of diffeomorphisms $V_t = e^{v_t}$ it suffices to find a family of automorphisms $\Psi_t : C^\infty M \longrightarrow C^\infty M$ smoothly depending on $t \in [0, a]$ such that $\Psi_t^{-1} \frac{d}{dt} \Psi_t = v_t$. This is the same as to solve the ordinary differential equation

$$\frac{df_t}{dt} = v_t(f_t) \qquad (*)$$

for any given f_0. Then $\Psi_t(f_0) := f_t$ clearly satisfies $\frac{d}{dt}\Psi_t(f_0) = v_t \Psi_t(f_0)$.

To finish the proof, **we need to show that a solution of (*) exists and is unique,** and **to prove that Ψ_t defined this way is an automorphism, that is, satisfies $\Psi_t(fg) = \Psi_t(f)\Psi_t(g)$.**

Existence and uniqueness of solutions of ODE

THEOREM: Consider the differential equation

$$\frac{df_t}{dt} = v_t(f_t), \qquad (*)$$

where $f_t \in C^\infty M$, and $t \in [0, a]$. Suppose that v_t has compact support. **Then (*) has a unique solution for each initial value f_0.**

Proof. Step 1: Since $[0, a]$ is compact, it suffices to solve (*) for sufficiently small values of t. Also, we may represent f_0 as a sum of functions with compact support in coordinate patches, using partition of unity. It remains to solve (*) when f_0 has compact support in $\mathbb{R}^n$.

Step 2: Existence and uniqueness of solutions of (*) follows from Peano and Picard-Lindelöf theorem. Recall that a function $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **Lipschitz** if $|\mu(x) - \mu(y)| < C|x - y|$ for all x, y. Let D be an open subset of $\mathbb{R} \times \mathbb{R}^n$, $f \in C^\infty D$, and

$$\frac{df_t}{dt} = v(t, f(t)) \qquad (**)$$

a continuous first-order differential equation defined on D. **(Peano) Then for every initial value f_0 there exists a solution of (**) defined on a small interval $[0, \varepsilon]$.** Moreover **(Picard-Lindelöf) the solution is unique if v is Lipschitz**. Notice that v is Lipschitz on any compact set if it is smooth.

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Derivations and automorphisms

To finish Theorem 1, **it would suffice to show that the map $f_0 \stackrel{\Psi_t}{\longrightarrow} f_t$ obtained as a solution of $\frac{df_t}{dt} = v_t(f_t)$ is multiplicative:** $\Psi_t(fg) = \Psi_t(f)\Psi_t(g)$. From the definition of Ψ_t it follows

$$\frac{d}{dt}\Psi_t(fg) = v_t(f_t)g_t + f_t v(g_t)$$

and

$$\frac{d}{dt}\Big(\Psi_t(f)\Psi_t(g)\Big) = v_t(f_t)g_t + f_t v(g_t)$$

Therefore, both $\Psi_t(fg)$ and $\Psi_t(f)\Psi_t(g)$ are solition of a differential equation $\frac{d}{dt}(\chi_t) = v_t(\chi_t)$ with the same initial value $\chi_0 = fg$. **They are equal by uniqueness of solutions.**

The same argument proves the following lemma **(prove it).**

LEMMA: Let v, v' be commuting vector fields. **Then the corresponding diffeomorphisms commute.** Moreover, $V_t(v') = v'$, **where V_t is the diffeomorphism flow associated with v.**

Distributions

DEFINITION: Distribution on a manifold is a sub-bundle $B \subset TM$

REMARK: Let $\Pi: TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that **$\Pi([x, y])$ is $C^\infty(M)$-linear as a function of x and y.**

DEFINITION: The map $[B, B] \longrightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

DEFINITION: A distribution is called **integrable**, or **holonomic**, or **involutive**, if its Frobenuus form vanishes.

Smooth submersions

DEFINITION: Let $\pi : M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and **$\pi|_U$ is a projection of $U \cong V$ tp $W \subset M'$ along U.**

EXERCISE: Deduce this result from the inverse function theorem.

EXERCISE: ("Ehresmann's fibration theorem")
Let $\pi : M \longrightarrow M'$ be a smooth submersion of compact manifolds. Prove that π is a locally trivial fibration.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi : M \longrightarrow M'$ is the kernel of $D\pi$.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion and $T_\pi M \subset TM$ the vertical tangent space. **Then $T_\pi M$ is an involutive subbundle.**

Proof: $D_\pi([X,Y]) = [D_\pi(X), D_\pi(Y)] = 0$ for any $X, Y \in \ker D_\pi$. ∎

Frobenius theorem (statement)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication **"$B = T_\pi M$"** $\Rightarrow$ **"Frobenius form vanishes"** was proven above.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

Frobenius theorem: existence of integral submanifolds

REMARK: To prove the Frobenius theorem for $B \subset TM$, **it suffices to show that each point is contained in an interal submanifold.** In this case, the smooth submersion $U \xrightarrow{\pi} V$ is a projection to the leaf space of the distribution.

REMARK: When B is 1-dimensional (in this case one says that B **has rank 1**, denoted $\operatorname{rk} B = 1$), **Frobenius theorem follows from existence of the diffeomorphism flow associated with a vector field.** Indeed, locally we may assume that B admits a non-degenerate section v. Let $V_t : M \times \mathbb{R} \longrightarrow M$ be the corresponding flow of diffeomorphisms. Then $Z_m := V_t(\{m\} \times \mathbb{R}$ is tangent to v everywhere, hence it is a 1-dimensional manifold immersed in M. Clearly, Z_m is a leaf this distribution. **Since B is a tangent to a foliation, it is integrable.**

Basic sub-bundles (1)

DEFINITION: Let $B \subset TM$ be an involutive sub-bundle. A sub-bundle $F \subset TM$ is called **basic** for B if $F \supset B$ and for all $b \in B, b' \in F$, one has $[b, b'] \in F$.

REMARK: One should think of basic sub-bundles as of **sub-bundles preserved by all diffeomorphisms obtaned from exponentiation of a vector field $v \in B$.**

LEMMA: Let $B \subset TM$ be an integrable distribution, $\pi : M \longrightarrow M_1$ projection to the leaf space of B, and $F \supset B$ a sub-bundle of TM containing B. Then the following conditions are equivalent: **(a) F is basic for B.**
(b) There exists a sub-bundle $F_1 \subset TM_1$ such that $\pi^{-1}(F_1) = F$.

Proof: Next slide.

Basic sub-bundles (2)

LEMMA: Let $B \subset TM$ be an integrable distribution, $\pi: M \longrightarrow M_1$ projection to the leaf space of B, and $F \supset B$ a sub-bundle of TM containing B. Then the following conditions are equivalent: **(a) F is basic for B.**
(b) There exists a sub-bundle $F_1 \subset TM_1$ such that $\pi^{-1}F_1 = F$.

Proof. Step 1: Consider coordinates $x_1, ..., x_n$ on M such that $x_{k+1} = \pi^*(x'_{k+1}, ..., x_n = \pi^*(x_n)$, where $x'_i, i = k+1, k+2, ..., n$ are coordiantes on M_1, and $\frac{d}{dx_1}, ..., \frac{d}{dx_k}$ generate B. Locally such coordinates always exist, because B is integrable. Denote by G a subgroup of Diff(M) obtained by exponents of $\frac{d}{dx_1}, ..., \frac{d}{dx_k}$. Since $[b, b'] \subset F$, the corresponding diffeomorphisms commute. **Therefore, F is a G-invariant sub-bundle of TM.**

Step 2: Any G-invariant sub-bundle $F \supset B$ is obtained as $\pi^{-1}(F_1)$ for some sub-bundle $F_1 \subset TM_1 = M/G$. Indeed, since the action of G_1 is free, the bundle F is generated over $C^\infty M$ by G-invariant sections. However, any G-invariant bundle F containing B is generated by G-invariant sections, which can be lifted from M/G **(check this)**.

Step 3: Conversely, if F is lifted from $M_1 = M/G$, if is G-invariant, hence $e^{tb}(b') \subset F$, and this gives $[b, b'] \subset F$ **(check this)**. ■

Frobenius theorem (proof)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

Proof. Step 1: Consider a rank 1 sub-bundle $B_1 \subset B$. Using the diffeomorphism flow as above, we prove that B_1 is integrable. Since $[B_1, B] \subset B$, the bundle B is basic with respect to B_1. **Therefore, $B = \pi^{-1}(B')$ for some $B' \subset TM_1$, where M_1 is the leaf space of B_1.**

Step 2: Let $\pi : M \longrightarrow M_1$ be the projection to the leaf space. Then $B = \pi^{-1}(B')$, where $\operatorname{rk} B' = \operatorname{rk} B - 1$. Using induction in $\operatorname{rk} B$, we can assume that B' is integrable. Let $\pi_0 : M_1 \longrightarrow M_0$ be the projection to the leaf space of B', defined locally in M. **Then $\pi \circ \pi_0 : M \longrightarrow M_0$ is the projection to the leaf space of B.** ■