Differential geometry

lecture 8: Grassmann algebra

Misha Verbitsky

Université Libre de Bruxelles

December 5, 2016

Bilinear maps

DEFINITION: Let U, V, W be vector spaces over k. A map $U \times V \stackrel{\mu}{\to} W$, $u, v \mapsto \mu(u, v)$ is called **bilinear** if for all $u \in U$ and $v \in V$ the maps $\mu(u, \cdot) : V \to W \quad \mu(\cdot, v) : U \to W$ are linear.

REMARK: Clearly, a linear combination of bilinear maps is bilinear. Then **the space** Bil$(U \times V, W)$ **of bilinear maps** $U \times V \to W$ **is a vector space**.

DEFINITION: Bilinear form on V is a bilinear map $V \times V \stackrel{\mu}{\to} k$. **Bilinear symmetric form** is a form which satisfies $\mu(x, y) = \mu(y, x)$ for all $x, y \in V$. **Bilinear anti-symmetric formm** or **bilinear skew-symmetric form** is a form which satisfies $\mu(x, y) = -\mu(y, x)$. We denote the first by $\operatorname{Sym}^2 V^*$, and the second by $\Lambda^2 V^*$.

Symmetrization and antisymmetrization

DEFINITION: Consider the **symmetrization map** $\operatorname{Sym}(\mu)(x,y) = \frac{1}{2}(\mu(x,y)+\mu(y,x))$ from $\operatorname{Bil}(V\times V,k)$ to $\operatorname{Sym}^2(V^*)$ and **anti-symmetrization map** $\operatorname{Alt}(\mu)(x,y) = \frac{1}{2}(\mu(x,y)-\mu(y,x))$ from $\operatorname{Bil}(V\times V,k)$ to $\Lambda^2(V^*)$.

EXERCISE: Let A, B be vector subspaces in a vector space C. **Recall that** $C = A\oplus B$ means that $A\cap B = 0$ and C is generated by A, B. Prove that the map

$$\operatorname{Sym}\oplus\operatorname{Alt}: \ \operatorname{Bil}(V\times V,k)\to \operatorname{Sym}^2 V^*\oplus\Lambda^2 V^*$$

is an isomorphism.

EXERCISE: Let V be n-dimensional. **Find the dimension of** $\operatorname{Sym}^2 V^*$ **and** $\Lambda^2 V^*$**.**

Tensor product

DEFINITION: Let S be a set. Define **vector space, freely generated by** S as the space of functions $\psi : S \longrightarrow k$ which are equal zero outside of a finite subset $\operatorname{Sup}_\psi \subset S$.

DEFINITION: Let V, V' be vector spaces over k, and W a vector space freely generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subspace generated by combinations $av \otimes v' - v \otimes av'$, $a(v \otimes v') - (av) \otimes v'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$, where $a \in k$. Define **the tensor product** $V \otimes_k V'$ as a quotient vector space W/W_1.

PROPOSITION: For any vector spaces V, V', R, there is a natural identification $\operatorname{Hom}(V \otimes_k V', R) = \operatorname{Bil}(V \times V', R)$.

Proof: Clearly, any bilinear map $\rho \in \operatorname{Bil}(V \times V', R)$ defines a linear map $\tilde{\rho} : W \longrightarrow R$, and $\tilde{\rho}$ vanishes on W_1. This gives a map $\operatorname{Bil}(V \times V', R) \longrightarrow \operatorname{Hom}(V \otimes_k V', R)$. Inverse map takes $\tau \in \operatorname{Hom}(V \otimes_k V', R)$ and interprets it as a bilinear map in $\operatorname{Bil}(V \times V', R)$. ■

COROLLARY: For finite-dimensional V, V', one has $V \otimes_k V' = \operatorname{Bil}(V \times V', k)^*$.

Dimension of of tensor product

CLAIM: Dimension of $\operatorname{Bil}(V \times V', k)$ is equal to $\dim V \dim V'$.

Proof. Step 1: Let $\{\lambda_i\}$ be a basis in V^* and $\{\lambda'_i\}$ a basis in V'. Denote by $\{v_i\}$ $\{v'_i\}$ the dual basis in V, V'. Then $\lambda_i \lambda'_j$ can be interpreted as vectors in $\operatorname{Bil}(V \times V', k)$. These vectors are clearly linearly independent: indeed

$$\sum_{i,j} a_{ij} \lambda_i \lambda'_j (v_p, v'_q) = a_{pq}.$$

This gives $\dim \operatorname{Bil}(V \times V', k) \geqslant \dim V \dim V'$.

Step 2: On the other hand, $\dim V \otimes V' \leqslant \dim V \dim V'$, because it is generated by $v_p \otimes v_q$, hence $\dim \operatorname{Bil}(V \times V', k) \leqslant \dim V \dim V'$. ■

COROLLARY: Let $\{x_i\}$ and $\{y_i\}$ be bases in V, W. **Then $\{x_i \otimes y_j\}$ is a basis in $V \otimes_k W$.** ■

Algebra over a field (reminder)

Fix a ground field k. Recall that a map $(V_1 \times V_2) \xrightarrow{\mu} V_3$ of vector spaces is called **bilinear** of for any $v_1 \in V_1$, $v_2 \in V_2$, the maps $\mu(v_1, \cdot) : V_2 \longrightarrow V_3$, $\mu(\cdot, v_2) : V_1 \longrightarrow V_3$ (one element is fixed) is k-linear.

To express this, we use the tensor product sign, and write $\mu : V_1 \otimes V_2 \longrightarrow V_3$.

DEFINITION: Let A be a vector space over k, and $\mu : A \otimes A \longrightarrow A$ a bilinear map (called **"multiplication"**). The pair (A, μ) is called **algebra over a field k** if μ is **associative**: $\mu(a_1, \mu(a_2, a_3)) = \mu(\mu(a_1, a_2), a_3))$. The product in algebra is written as $a \cdot b$ or ab. If, in addition, there is an element $1 \in A$ such that $\mu(1, a) = \mu(a, 1) = a$ for all $a \in A$, this element is called **unity**, a and A **an algebra with unity**.

DEFINITION: A **homomorphism** of algebras $r : A \longrightarrow A'$ is a linear map which is compatible with a product. **Isomorphism** of algebras is an invertible homomorphism. **Subalgebra** of an algebra A is a vector subspace which is closed under multiplication.

Algebra of polylinear forms

DEFINITION: Let V be a vector space over a field k. A **polylinear n-form**, or **n-linear form** φ on V is a map

$$\varphi: \underbrace{V \times V \times V \times \cdots \times V}_{n \text{ times}} \longrightarrow k,$$

linear in each argument. We write this as $\varphi: V \otimes V \otimes V \otimes \cdots \otimes V \longrightarrow k$. It is convenient to denote the space of n-linear forms as $(V^*)^{\otimes^n}$. In this notation, $(V^*)^{\otimes^0}$ is k (the ground field).

DEFINITION: Given polylinear j- and i-forms φ and ψ on V, the map $\varphi \otimes \psi$: $\underbrace{V \times V \times V \times \ldots}_{i+j} \longrightarrow k$ is given by

$$(\varphi \otimes \psi)(v_1, v_2, \ldots, v_{i+j}) = \varphi(v_1, \ldots, v_i)\varphi(v_{i+1}, \ldots, v_{i+j})$$

is clearly polylinear. This gives a multiplicative structure on the space $\bigoplus_{i=0}^{\infty}(V^*)^{\otimes^i}$, which is clearly associative. We call $\bigoplus_{i=0}^{\infty}(V^*)^{\otimes^i}$ the **algebra of polylinear forms.**

Tensor algebra

Let $V \times W$ map to $V \otimes W$ by putting v, w to $v \otimes w$. Clearly, this map is bilinear. Similarly, one has a bilinear map $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes m+n}$ putting $x_1 \otimes ... \otimes x_n, y_1 \otimes ... \otimes y_n$ to $x_1 \otimes ... \otimes x_n \otimes y_1 \otimes ... \otimes y_n$.

DEFINITION: Let V be a vector space over k. **Tensor algebra**, or **free algebra generated by V** is $T(V) := \bigoplus_i V^{\otimes^i}$ equipped with the multiplicative structure defined above,

EXERCISE: Prove that $T(V^*)$ **is the algebra of polylinear forms on V defined above.**

REMARK: If $x_1, ..., x_r$ is a basis in V, then **the basis in $V^{\otimes^n}$ is formed by all different monomials** of the form $x_{i_1} \otimes x_{i_2} \otimes ... \otimes x_{i_n}$.

Universal property of tensor algebra

CLAIM: Let $\varphi: V \longrightarrow A$ be a linear map from a vector space V to an algebra A (with unit). **Then φ can be uniquely extended to a homomorphism $\Phi: T(V) \longrightarrow A$** respecting a unit.

Proof. Step 1: Uniqueness is clear: indeed, $T(V)$ **is multiplicatively generated by** V (and unit).

Step 2: The vector space $T(V)$ is a quotient of the space $T_f(V)$ freely generated by the symbols $x_1 \otimes x_2 \otimes ... \otimes x_n$, $x_i \in V$ by the space $T_b(V)$ generated by "bilinear relations" of type

$$x_1 \otimes x_2 \otimes ... \otimes (x_i + x_i') \otimes ... \otimes x_n = x_1 \otimes x_2 \otimes ... \otimes x_i \otimes ... \otimes x_n + x_1 \otimes x_2 \otimes ... \otimes x_i' \otimes ... \otimes x_n.$$

and

$$x_1 \otimes x_2 \otimes ... \otimes a x_i \otimes ... \otimes x_n = a \cdot x_1 \otimes x_2 \otimes ... \otimes x_i \otimes ... \otimes x_n$$

The map φ is extended to $T_f(V)$ by putting

$$\Phi(x_1 \otimes x_2 \otimes ... \otimes x_n) = \varphi(x_1)\varphi(x_2)...\varphi(x_n).$$

This map vanishes on $T_w(V)$, because the product map $A \otimes A \otimes ... \otimes A \longrightarrow A$ satisfies the bilinear relations. ■

Two-sided ideals

DEFINITION: Let A be an algebra and $J \subset A$ its subspace. Then J is caled **left ideal** if for all $a \in A, j \in J$, one has $ja \in J$, and **right ideal** if one has $aj \in J$. J is called **two-sided ideal** if is is both right and left ideal.

REMARK: Let $J \subset A$ be a two-sided ideal, $x, y \in A/J$ some vectors, and $\tilde{x}, \tilde{y}$ Define the product $xy \in A/J$ by putting $x \cdot y$ to the class represented by $\tilde{x}, \tilde{y}$. Since $ja \in J$ and $aj \in J$, **this gives a bilinear map** $A/J \otimes A/J \longrightarrow A/J$, defining an associative multiplicative structure on A/J.

CLAIM: In these assumptions, A/J **is an algebra,** with the product defined as above.

EXERCISE: Prove it.

Algebra defined by generators and relations

DEFINITION: Let V be a vector space over k **("the space of generators")**, and $W \subset T(V)$ another vector space **("the space of relations")**. Consider a quotient A of $T(V)$ by the subspace $T(V)WT(V)$ generated by the vectors $v \otimes w \otimes v'$, where $w \in W$ and $v, v' \in T(V)$.

CLAIM: There is a natural product structure on the space $A := \frac{T(V)}{T(V)WT(V)}$.

Proof: $T(V)WT(V)$ is a 2-sided ideal. ■

DEFINITION: In this situation, we say that A **is an algebra defined by generators and relations**.

EXERCISE: Prove that any algebra can be defined by generators and relations.

DEFINITION: An algebra is called **finitely generated** if it can be defined by generators and relations, and the space of generators is finitely-dimensional. An algebra is called **finitely presented** if the space W of relations is finitely-dimensional.

Graded algebras

DEFINITION: An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead of $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always in A^0, so that it is non-empty.

EXAMPLE: The tensor algebra $T(V)$ and the polynomial algebra $\operatorname{Sym}^*(V)$ are obviously graded.

DEFINITION: A subspace $W \subset A^*$ of a graded algebra is called **graded** if W is a direct sum of components $W^i \subset A^i$.

EXERCISE: Let $W \subset T(V)$ be a graded subspace. Prove that then **the algebra generated by V with relation space W is also graded.**

Symmetric algebra

DEFINITION: Consider a subspace $W \subset V \otimes V$ generated by vectors $x \otimes y - y \otimes x$. Then the algebra $\operatorname{Sym}^*(V) := \frac{T(V)}{T(V)WT(V)}$ is commutative **(check this)**. Since W is a graded subspace, $\operatorname{Sym}^*(V)$ is a graded algebra.

CLAIM: For any commutative algebra A over k and any linear map $\varphi : V \longrightarrow A$, **$\varphi$ can be uniquely extended to an algebra homomorphism $\Phi : \operatorname{Sym}^*(V) \longrightarrow A$.**

Proof. Step 1: Clearly, φ can be extended to a homomorphism $\Phi_t : T^*V \longrightarrow A$ from the tensor algebra T^*V.

Step 2: Since A is commutative, $\Phi_t(xy) = \Phi_t(yx)$. Therefore, Φ_t vanishes on the ideal $T(V)WT(V)$. ■

COROLLARY: Let V be a vector space over k with basis $x_1, ..., x_n$. Then **$\operatorname{Sym}^* V$ is isomorphic to the polynomial algebra $k[x_1, ..., x_n]$.**

Proof: Consider the linear map $V = \langle x_1, ..., x_n \rangle \longrightarrow k[x_1, ..., x_n]$ mapping a vector to the corresponding homogeneous degree 1 polynomial. This map can be extended to a homomorphism $\operatorname{Sym}^* V \longrightarrow k[x_1, ..., x_n]$ as shown above. Inverse map takes $x_i \in k[x_1, ..., x_n]$ to the corresponding vector $x_i \in V$. ■

The Grassmann algebra

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a subspace generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i \in \mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i \in \mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

Signature of a permutation

DEFINITION: The group Σ_n acts on the polynomial ring $P[x_1,...,x_n]$ by permutation of variables. Consider the polynomial $P(x_1,...,x_n) := \prod_{i<j}(x_i - x_j)$. Clearly, for any permutation $\sigma \in \Sigma_n$, we have $\sigma(P) = \pm P$. **This defines a homomorphism $\operatorname{sign}: \Sigma_n \longrightarrow \{\pm 1\}$, with $\sigma(P) = \operatorname{sign}(\sigma)P$.**

REMARK: This homomorphism maps a product of odd number of transpositions to -1 and a product of even number of transpositions to 1.

DEFINITION: The number $\operatorname{sign}(\sigma)$ is called **signature** of a permutation σ. Permutation σ is called **odd** if $\operatorname{sign}(I) = -1$ and **even** if $\operatorname{sign}(\sigma) = 1$. For odd permutation σ we write $\tilde{\sigma} := 1$, for even permutation, $\tilde{\sigma} = 0$.

Antisymmetric tensors

DEFINITION: Let $V^{\otimes^n}$ be n-th product of V with itself, equipped with the natural symmetric group Σ_n-action exchanging the tensor components. A tensor $\psi \in V^{\otimes^n}$ is called **antisymmetric** if for any permutation $\sigma \in \Sigma_n$ we have $\sigma(\psi) = (-1)^{\tilde\sigma}\psi$, and **symmetric** if $\sigma(\psi) = \psi$. We denote the space of all antisymmetric tensors by $\tilde\Lambda^n V$ and the space of symmetric tensors by $\widetilde{\operatorname{Sym}}^n V$.

Theorem 1: Let V be a vector space, $\Lambda^n V$ the n-th component of its Grassmann algebra, and $\operatorname{Sym}^n V$ the n-th component of its symmetric algebra. Then **$\Lambda^n V$ is naturally identified with $\tilde\Lambda^n V$, and $\operatorname{Sym}^n V$ with $\widetilde{\operatorname{Sym}}^n V$:** the projection from $V^{\otimes n}$ to $\Lambda^n V$ (or $\operatorname{Sym}^n V$) induces an isomorphism from $\tilde\Lambda^n V$ (or $\widetilde{\operatorname{Sym}}^n V$) to $\Lambda^n V$ (or $\operatorname{Sym}^n V$).

Antisymmetrization

The natural map from $\Lambda^n V$ to $\tilde{\Lambda}^n V$ is given by **antisymmetrization**

$$\mathrm{Alt}(x_1 \otimes ... \otimes x_n) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{\tilde{\sigma}} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}.$$

Its properties:

1. Clearly, $\operatorname{im} \mathrm{Alt}$ lies in the space of antisymmetric tensors, and $\mathrm{Alt}(\eta - (-1)^{\tilde{\sigma}}\eta) = 0$, hence Alt **defines a map from Grassmann algebra to the space of antisymmetric tensors.**

2. The natural **projection from antisymmetric tensors to the Grassmann algebra is inverse to** Alt**.**

Grassmann algebra: dimension of components

REMARK: For linearly independent vectors $x_1, ..., x_k$, the antisymmetrization $x_1 \wedge x_2 \wedge ... \wedge x_k := \frac{1}{k!} \sum (-1)^{\tilde\sigma} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ is non-trivial. Indeed, the monomials $x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ are linearly independent in $V^{\otimes k}$. **This implies** $\dim \Lambda^k V \geqslant \binom{\dim V}{k}$.

CLAIM: $\dim \Lambda^k V = \binom{\dim V}{k}$**, and** $\dim \Lambda^* V = 2^{\dim V}$**.**

Proof: Let $x_1, ..., x_n$ be a basis in V. Then **the space $\Lambda^k V$ is generated by antisymmetric tensors $x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_k}$, $i_1 < i_2 < ... < i_k$, which are all linearly independent.** ∎

Grassmann algebra and determinant

REMARK: Let W be a one-dimensional vector space over k. **Then** End W **is naturally isomorphic to k.**

REMARK: Let $A \in \text{End}(V)$ be a linear endomorphism of a vector space V. Then **the action of A on $V \cong \Lambda^1 V$ is uniquely extended to a multiplicative endomorphism of the algebra $\Lambda^* V$.**

DEFINITION: Let V be a d-dimensional vector space and $A \in \text{End}(V)$. Consider the induced endomorphism of the space of determinant vectors $\Lambda^d(V)$ denoted as $\det A \in \text{End}(\Lambda^d(V))$. Since $\Lambda^d(V)$ is 1-dimensional, the space $\text{End}(\Lambda^d(V))$ is naturally identified with k. **This allows to consider** $\det A$ **as a number, that is, an element of k.** This number is called **determinant** of A.

REMARK: From this definition it is clear that det **defines a homomorphism from the group $GL(V)$ if invertible matrices to the multiplicative group k^* of the ground field.**