Differential geometry

lecture 9: de Rham differential

Misha Verbitsky

Université Libre de Bruxelles

December 12, 2016

Bilinear maps (reminder)

DEFINITION: Let U, V, W be vector spaces over k. A map $U \times V \xrightarrow{\mu} W$, $u, v \mapsto \mu(u,v)$ is called **bilinear** if for all $u \in U$ and $v \in V$ the maps $\mu(u,\cdot) : V \to W$ $\mu(\cdot,v) : U \to W$ are linear.

REMARK: Clearly, a linear combination of bilinear maps is bilinear. Then **the space** $\mathrm{Bil}(U \times V, W)$ **of bilinear maps** $U \times V \to W$ **is a vector space**.

DEFINITION: **Bilinear form** on V is a bilinear map $V \times V \xrightarrow{\mu} k$. **Bilinear symmetric form** is a form which satisfies $\mu(x,y) = \mu(y,x)$ for all $x, y \in V$. **Bilinear anti-symmetric form**m or **bilinear skew-symmetric form** is a form which satisfies $\mu(x,y) = -\mu(y,x)$. We denote the first by $\mathrm{Sym}^2 V^*$, and the second by $\Lambda^2 V^*$.

Tensor product (reminder)

DEFINITION: Let S be a set. Define **vector space, freely generated by S** as the space of functions $\psi: S \longrightarrow k$ which are equal zero outside of a finite subset $\operatorname{Sup}_\psi \subset S$.

DEFINITION: Let V, V' be vector spaces over k, and W a vector space freely generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subspace generated by combinations $av \otimes v' - v \otimes av'$, $a(v \otimes v') - (av) \otimes v'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v_1' + v_2') - v \otimes v_1' - v \otimes v_2'$, where $a \in k$. Define **the tensor product** $V \otimes_k V'$ as a quotient vector space W/W_1.

PROPOSITION: For any vector spaces V, V', R, there is a natural identification $\operatorname{Hom}(V \otimes_k V', R) = \operatorname{Bil}(V \times V', R)$.

Proof: Clearly, any bilinear map $\rho \in \operatorname{Bil}(V \times V', R)$ defines a linear map $\tilde\rho: W \longrightarrow R$, and $\tilde\rho$ vanishes on W_1. This gives a map $\operatorname{Bil}(V \times V', R) \longrightarrow \operatorname{Hom}(V \otimes_k V', R)$. Inverse map takes $\tau \in \operatorname{Hom}(V \otimes_k V', R)$ and interprets it as a bilinear map in $\operatorname{Bil}(V \times V', R)$. ■

COROLLARY: For finite-dimensional V, V', one has $V \otimes_k V' = \operatorname{Bil}(V \times V', k)^*$.

Dimension of of tensor product (reminder)

CLAIM: Dimension of $\operatorname{Bil}(V \times V', k)$ **is equal to** $\dim V \dim V'$**.**

Proof. Step 1: Let $\{\lambda_i\}$ be a basis in V^* and $\{\lambda'_i\}$ a basis in V'. Denote by $\{v_i\}$ $\{v'_i\}$ the dual basis in V, V'. Then $\lambda_i \lambda'_j$ can be interpreted as vectors in $\operatorname{Bil}(V \times V', k)$. These vectors are clearly linearly independent: indeed

$$\sum_{i,j} a_{ij} \lambda_i \lambda'_j (v_p, v'_q) = a_{pq}.$$

This gives $\dim \operatorname{Bil}(V \times V', k) \geqslant \dim V \dim V'$.

Step 2: On the other hand, $\dim V \otimes V' \leqslant \dim V \dim V'$, because it is generated by $v_p \otimes v_q$, hence $\dim \operatorname{Bil}(V \times V', k) \leqslant \dim V \dim V'$. ■

COROLLARY: Let $\{x_i\}$ and $\{y_i\}$ be bases in V, W. **Then** $\{x_i \otimes y_j\}$ **is a basis in** $V \otimes_k W$**.** ■

Algebra over a field (reminder)

Fix a ground field k. Recall that a map $(V_1 \times V_2) \xrightarrow{\mu} V_3$ of vector spaces is called **bilinear** of for any $v_1 \in V_1$, $v_2 \in V_2$, the maps $\mu(v_1, \cdot): V_2 \longrightarrow V_3$, $\mu(\cdot, v_2): V_1 \longrightarrow V_3$ (one element is fixed) is k-linear.

To express this, we use the tensor product sign, and write $\mu: V_1 \otimes V_2 \longrightarrow V_3$.

DEFINITION: Let A be a vector space over k, and $\mu: A \otimes A \longrightarrow A$ a bilinear map (called **"multiplication"**). The pair (A, μ) is called **algebra over a field k** if μ is **associative**: $\mu(a_1, \mu(a_2, a_3)) = \mu(\mu(a_1, a_2), a_3))$. The product in algebra is written as $a \cdot b$ or ab. If, in addition, there is an element $1 \in A$ such that $\mu(1, a) = \mu(a, 1) = a$ for all $a \in A$, this element is called **unity**, a and A **an algebra with unity**.

DEFINITION: A **homomorphism** of algebras $r: A \longrightarrow A'$ is a linear map which is compatible with a product. **Isomorphism** of algebras is an invertible homomorphism. **Subalgebra** of an algebra A is a vector subspace which is closed under multiplication.

Tensor algebra (reminder)

Let $V \times W$ map to $V \otimes W$ by putting v, w to $v \otimes w$. Clearly, this map is bilinear. Similarly, one has a bilinear map $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes m+n}$ putting $x_1 \otimes ... \otimes x_n, y_1 \otimes ... \otimes y_n$ to $x_1 \otimes ... \otimes x_n \otimes y_1 \otimes ... \otimes y_n$.

DEFINITION: Let V be a vector space over k. **Tensor algebra**, or **free algebra generated by V** is $T(V) := \bigoplus_i V^{\otimes^i}$ equipped with the multiplicative structure defined above,

EXERCISE: Prove that $T(V^*)$ **is the algebra of polylinear forms on V defined above.**

REMARK: If $x_1, ..., x_r$ is a basis in V, then **the basis in $V^{\otimes^n}$ is formed by all different monomials** of the form $x_{i_1} \otimes x_{i_2} \otimes ... \otimes x_{i_n}$.

Two-sided ideals (reminder)

DEFINITION: Let A be an algebra and $J \subset A$ its subspace. Then J is caled **left ideal** if for all $a \in A, j \in J$, one has $ja \in J$, and **right ideal** if one has $aj \in J$. J is called **two-sided ideal** if is is both right and left ideal.

REMARK: Let $J \subset A$ be a two-sided ideal, $x, y \in A/J$ some vectors, and $\tilde{x}, \tilde{y}$ Define the product $xy \in A/J$ by putting $x \cdot y$ to the class represented by $\tilde{x}, \tilde{y}$. Since $ja \in J$ and $aj \in J$, **this gives a bilinear map** $A/J \otimes A/J \longrightarrow A/J$, defining an associative multiplicative structure on A/J.

CLAIM: In these assumptions, A/J **is an algebra,** with the product defined as above.

EXERCISE: Prove it.

Algebra defined by generators and relations (reminder)

DEFINITION: Let V be a vector space over k **("the space of generators")**, and $W \subset T(V)$ another vector space **("the space of relations")**. Consider a quotient A of $T(V)$ by the subspace $T(V)WT(V)$ generated by the vectors $v \otimes w \otimes v'$, where $w \in W$ and $v, v' \in T(V)$.

CLAIM: **There is a natural product structure on the space $A := \frac{T(V)}{T(V)WT(V)}$.**

Proof: $T(V)WT(V)$ is a 2-sided ideal. ■

DEFINITION: In this situation, we say that A **is an algebra defined by generators and relations**.

EXERCISE: **Prove that any algebra can be defined by generators and relations.**

DEFINITION: An algebra is called **finitely generated** if it can be defined by generators and relations, and the space of generators is finitely-dimensional. An algebra is called **finitely presented** if the space W of relations is finitely-dimensional.

Graded algebras (reminder)

DEFINITION: An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead of $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always in A^0, so that it is non-empty.

EXAMPLE: The tensor algebra $T(V)$ and the polynomial algebra $\operatorname{Sym}^*(V)$ are obviously graded.

DEFINITION: A subspace $W \subset A^*$ of a graded algebra is called **graded** if W is a direct sum of components $W^i \subset A^i$.

EXERCISE: Let $W \subset T(V)$ be a graded subspace. Prove that then **the algebra generated by V with relation space W is also graded.**

The Grassmann algebra (reminder)

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a subspace generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde x$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde x \tilde y} y \wedge x$.

Antisymmetric tensors (reminder)

DEFINITION: Let $V^{\otimes^n}$ be n-th product of V with itself, equipped with the natural symmetric group Σ_n-action exchanging the tensor components. A tensor $\psi \in V^{\otimes^n}$ is called **antisymmetric** if for any permutation $\sigma \in \Sigma_n$ we have $\sigma(\psi) = (-1)^{\tilde{\sigma}}\psi$, and **symmetric** if $\sigma(\psi) = \psi$. We denote the space of all antisymmetric tensors by $\tilde{\Lambda}^n V$ and the space of symmetric tensors by $\widetilde{\operatorname{Sym}}^n V$.

Theorem 1: Let V be a vector space, $\Lambda^n V$ the n-th component of its Grassmann algebra, and $\operatorname{Sym}^n V$ the n-th component of its symmetric algebra. Then **$\Lambda^n V$ is naturally identified with $\tilde{\Lambda}^n V$, and $\operatorname{Sym}^n V$ with $\widetilde{\operatorname{Sym}}^n V$:** the projection from $V^{\otimes n}$ to $\Lambda^n V$ (or $\operatorname{Sym}^n V$) induces an isomorphism from $\tilde{\Lambda}^n V$ (or $\widetilde{\operatorname{Sym}}^n V$) to $\Lambda^n V$ (or $\operatorname{Sym}^n V$).

Antisymmetrization (reminder)

The natural map from $\Lambda^n V$ to $\tilde{\Lambda}^n V$ is given by **antisymmetrization**

$$\operatorname{Alt}(x_1 \otimes ... \otimes x_n) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{\tilde{\sigma}} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}.$$

Its properties:

1. Clearly, $\operatorname{im}\operatorname{Alt}$ lies in the space of antisymmetric tensors, and $\operatorname{Alt}(\eta - (-1)^{\tilde{\sigma}}\eta) = 0$, hence Alt **defines a map from Grassmann algebra to the space of antisymmetric tensors.**

2. The natural **projection from antisymmetric tensors to the Grassmann algebra is inverse to** Alt**.**

Grassmann algebra: dimension of components (reminder)

REMARK: For linearly independent vectors $x_1, ..., x_k$, the antisymmetrization $x_1 \wedge x_2 \wedge ... \wedge x_k := \frac{1}{k!}\sum(-1)^{\tilde\sigma} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ is non-trivial. Indeed, the monomials $x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ are linearly independent in $V^{\otimes k}$. **This implies** $\dim \Lambda^k V \geqslant \binom{\dim V}{k}$.

CLAIM: $\dim \Lambda^k V = \binom{\dim V}{k}$**, and** $\dim \Lambda^* V = 2^{\dim V}$**.**

Proof: Let $x_1, ..., x_n$ be a basis in V. Then **the space $\Lambda^k V$ is generated by antisymmetric tensors $x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_k}$, $i_1 < i_2 < ... < i_k$, which are all linearly independent.** ■

Tensor product of vector bundles

DEFINITION: Let V, V' be R-modules, W a free abelian group generated by $v\otimes v'$, with $v\in V, v'\in V'$, and $W_1\subset W$ a subgroup generated by combinations $rv\otimes v' - v\otimes rv'$, $(v_1+v_2)\otimes v' - v_1\otimes v' - v_2\otimes v'$ and $v\otimes(v_1'+v_2') - v\otimes v_1' - v\otimes v_2'$. Define **the tensor product** $V\otimes_R V'$ as a quotient group W/W_1.

EXERCISE: Show that $r\cdot v\otimes v'\mapsto (rv)\otimes v'$ **defines an R-module structure on $V\otimes_R V'$.**

REMARK: Let $\mathcal{F}$ be a sheaf of rings, and $\mathcal{B}_1$ and $\mathcal{B}_2$ be sheaves of locally free $(M,\mathcal{F})$-modules. **Then**

$$U\longrightarrow \mathcal{B}_1(U)\otimes_{\mathcal{F}(U)}\mathcal{B}_2(U)$$

is also a locally free sheaf of modules.

DEFINITION: **Tensor product** of vector bundles is a tensor product of the corresponding sheaves of modules.

EXERCISE: Let B and B' ve vector bundles on M, $B\big|_x$, $B'\big|_x$ their fibers, and $B\otimes_{C^\infty M} B'$ their tensor product. **Prove that** $B\otimes_{C^\infty M} B'\big|_x = B\big|_x\otimes_{\mathbb{R}} B'\big|_x$.

Cotangent bundle

DEFINITION: Let M be a smooth manifold, TM the tangent bundle, and $\Lambda^1 M = T^*M$ its dual bundle. It is called **cotangent bundle** of M. Sections of T^*M are called **1-forms** or **covectors** on M. For any $f \in C^\infty M$, consider a functional $TM \longrightarrow C^\infty M$ obtained by mapping $X \in TM$ to a derivation of f: $X \longrightarrow D_X(f)$. Since this map is linear in X, it defines a section $df \in T^*M$ called **the differential** of f.

CLAIM: $\Lambda^1 M$ is generated as a $C^\infty M$-module by $d(C^\infty M)$.

Proof: Locally in coordinates $x_1, ..., x_n$ this is clear, because the covectors $dx_1, ..., dx_n$ dive a basis in T^*M dual to the basis $\frac{d}{dx_1}, ..., \frac{d}{dx_n}$ in TM. ■

DEFINITION: Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^*M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

REMARK: $\Lambda^0 M = C^\infty M$.

De Rham algebra

DEFINITION: Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on V. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

DEFINITION: Let $\bigotimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^*M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x \in M$ **is identified with the Grassmann algebra $\Lambda^* T_x^* M$.** This identification is compatible with the Grassmann product.

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha \in \Lambda^* \mathbb{R}^n$ a monomial obtained as a product of several dt_i: $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$ $i_1 < i_2 < ... < i_k$. Then α is called **a coordinate monomial**.

De Rham differential

DEFINITION: De Rham differential $d: \Lambda^* M \longrightarrow \Lambda^{*+1} M$ is an $\mathbb{R}$-linear map satisfying the following conditions.

* For each $f \in \Lambda^0 M = C^\infty M$, $d(f) \in \Lambda^1 M$ is equal to the differential $df \in \Lambda^1 M$.
* **(Leibnitz rule)** $d(a \wedge b) = da \wedge b + (-1)^j a \wedge db$ for any $a \in \Lambda^i M, b \in \Lambda^j M$.
* $d^2 = 0$.

REMARK: A map on a graded algebra which satisfies the Leibnitz rule above is called **an odd derivation**.

REMARK: The following two lemmas are needed to prove uniqueness of de Rham differential.

LEMMA: Let $A = \bigoplus A^i$ be a graded algebra, $B \subset A$ a set of multiplicative generators, and $D_1, D_2: A \longrightarrow A$ two odd derivations which are equal on B. **Then $D_1 = D_2$.** ■

LEMMA: $\Lambda^* M$ is generated by $C^\infty M$ and $d(C^\infty M)$.

Proof: By definition, $\Lambda^* M$ is generated by $\Lambda^0 M = C^\infty M$ and $\Lambda^1 M$. However, $d(C^\infty M)$ generate $\Lambda^1 M$, as shown above. ■

De Rham differential: uniqueness and existence

THEOREM:
De Rham differential is uniquely determined by these axioms.

Proof: De Rham differential is an odd derivation. Its value on $C^\infty M$ is defined by the first axiom. On $d(C^\infty M)$ de Rham differential valishes, because $d^2 = 0$. ■

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, α_i coordinate monomials, and $\alpha := \sum f_i \alpha_i$. Define $d(\alpha) := \sum_i \sum_j \frac{df_i}{dt_j} dt_j \wedge \alpha_i$.

EXERCISE:
Check that d satisfies the properties of de Rham differential.

COROLLARY: De Rham differential exists on any smooth manifold.

Proof: Locally, de Rham differential d exists, as follows from the construction above. Since d is unique, it is compatible with restrictions. **This means that d defines a sheaf morphism.** Restricting this sheaf morphism to global sections, we obtain de Rham differential on $\Lambda^* M$. ■

Superalgebras

DEFINITION: Let $A^* = \oplus_{i\in\mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij} ba$ for all $a \in A^i, b \in A^j$.

EXAMPLE: Grassmann algebra $\Lambda^* V$ is clearly supercommutative.

DEFINITION: Let A^* be a graded commutative algebra, and $D: A^* \longrightarrow A^{*+i}$ be a map which shifts grading by i. It is called a **graded derivation**, if $D(ab) = D(a)b + (-1)^{ij} aD(b)$, for each $a \in A^j$.

REMARK: If i is even, graded derivation is a usual derivation. If it is even, it an odd derivation.

DEFINITION: Let M be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **convolution with a vector field** $i_X: \Lambda^i M \longrightarrow \Lambda^{i-1} M$, mapping an i-form α to an $(i-1)$-form $v_1, ..., v_{i-1} \longrightarrow \alpha(X, v_1, ..., v_{i-1})$

EXERCISE: Prove that i_X is an odd derivation.

Supercommutator

DEFINITION: Let A^* be a graded vector space, and $E: A^* \longrightarrow A^{*+i}$, $F: A^* \longrightarrow A^{*+j}$ operators shifting the grading by i, j. Define **the supercommutator** $\{E, F\} := EF - (-1)^{ij} FE$.

DEFINITION: An endomorphism of a graded vector space which shifts grading by i is called **even** if i is even, and **odd** otherwise.

EXERCISE: Prove that the supercommutator satisfies **graded Jacobi identity,**

$$\{E, \{F, G\}\} = \{\{E, F\}, G\} + (-1)^{\tilde{E}\tilde{F}} \{F, \{E, G\}\}$$

where $\tilde{E}$ and $\tilde{F}$ are 0 if E, F are even, and 1 otherwise.

REMARK: There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. **Each time when in commutative case two letters E, F are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde{E}\tilde{F}}$.**

EXERCISE: **Prove that a supercommutator of superderivations is again a superderivation.**

Pullback of a differential form

DEFINITION: Let $M \xrightarrow{\varphi} N$ be a morphism of smooth manifolds, and $\alpha \in \Lambda^i N$ be a differential form. Consider an i-form $\varphi^*\alpha$ taking value

$$\alpha\Big|_{\varphi(m)}(D_\varphi(x_1), ...D_\varphi(x_i))$$

on $x_1, ..., x_i \in T_m M$. It is called **the pullback of α**. If $M \xrightarrow{\varphi} N$ is a closed embedding, the form $\varphi^*\alpha$ is called **the restriction** of α to $M \hookrightarrow N$.

LEMMA: (*) Let $\Psi_1, \Psi_2 : \Lambda^* N \longrightarrow \Lambda^* M$ be two maps which satisfy graded Leibnitz identity, supercommutes with de Rham differential, and satisfy $\Psi_1|_{C^\infty M} = \Psi_2|_{C^\infty M}$. **Then $\Psi_1 = \Psi_2$.**

Proof: The algebra $\Lambda^* M$ is generated multiplicatively by $C^\infty M$ and $d(C^\infty M)$; restrictions of Ψ_i to these two spaces are equal. ■

CLAIM: Pullback commutes with the de Rham differential.

Proof: Let $d_1, d_2 : \Lambda^* N \longrightarrow \Lambda^{*+1} M$ be the maps $d_1 = \varphi^* \circ d$ and $d_2 = d \circ \varphi^*$. **These maps satisfy the Leibnitz identity, and they are equal on $C^\infty M$.** The super-commutator $\delta := \{d_i, d\}$ is equal to $d \circ \varphi^* \circ d$, it commutes with d, and equal 0 on functions. By Lemma (*), $\delta = 0$. Then d_i supercommutes with d. Applying Lemma (*) again, we obtain that $d_1 = d_2$. ■

Lie derivative

DEFINITION: Let B be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along v** if it satisfies the following conditions.

(1) On functions Lie_v is equal to a derivative along v. (2) $[\operatorname{Lie}_v, d] = 0$.

(3) Lie_v is a derivation of the de Rham algebra.

REMARK: The algebra $\Lambda^*(M)$ is generated by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. The restriction $\operatorname{Lie}_v \big|_{C^\infty M}$ is determined by the first axiom. On $d(C^\infty M)$ is also determined because $\operatorname{Lie}_v(df) = d(\operatorname{Lie}_v f)$. **Therefore, Lie_v is uniquely defined by these axioms.**

EXERCISE: Prove that $\{d, \{d, E\}\} = 0$ **for each** $E \in \operatorname{End}(\Lambda^* M)$.

THEOREM: (Cartan's formula) Let i_v be a convolution with a vector field, $i_v(\eta) = \eta(v, \cdot, \cdot, ..., \cdot)$ **Then $\{d, i_v\}$ is equal to the Lie derivative along v.**

Proof: $\{d, \{d, i_v\}\} = 0$ by the lemma above. A supercommutator of two graded derivations is a graded derivation. Finally, $\{d, i_v\}$ acts on functions as $i_v(df) = \langle v, df \rangle$. ■

Flow of diffeomorphisms

DEFINITION: Let $f: M \times [a,b] \longrightarrow M$ be a smooth map such that for all $t \in [a,b]$ the restriction $f_t := f\Big|_{M\times\{t\}} : M \longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f \in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c} : C^\infty M \longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^*\frac{dV_t}{dt}|_{t=c}f$. **Then $f \longrightarrow (V_t^{-1})^*\frac{d}{dt}V_t^* f$ is a derivation** (that is, a vector field).

Proof: $\frac{d}{dt}V_t^*(fg) = V_t^*(f)\frac{d}{dt}V_t^* g + \frac{d}{dt}V_t^* f V_t^*(g)$ by the Leignitz rule, giving

$$(V_t^{-1})^*\frac{d}{dt}V_t^*(fg) = f(V_t^{-1})^*\frac{d}{dt}V_t^* g + g(V_t^{-1})^*\frac{d}{dt}V_t^* f.$$

■

DEFINITION: The vector field $\frac{d}{dt}V_t|_{t=c}$ is called **a vector field tangent to a flow of diffeomorphisms V_t at $t = c$.**

Lie derivative and a flow of diffeomorphisms

DEFINITION: Let v_t be a vector field on M, smoothly depending on the "time parameter" $t \in [a,b]$, and $V: M \times [a,b] \longrightarrow M$ a flow of diffeomorphisms which satisfies $\frac{d}{dt}V_t = v_t$ for each $t \in [a,b]$, and $V_0 = \mathrm{Id}$. Then V_t is called **an exponent of v_t.**

CLAIM: Exponent of a vector field is unique; it exists when M is compact. This statement is called **"Picard-Lindelöf theorem"** or **"uniqueness and existence of solutions of ordinary differential equations"**.

PROPOSITION: Let v_t be a time-dependent vector field, $t \in [a,b]$, and V_t its exponent. For any $\alpha \in \Lambda^* M$, consider $V_t^*\alpha$ as a $\Lambda^* M$-valued function of t. **Then** $\mathrm{Lie}_{v_0}(\alpha) = \frac{d}{dt}|_{t=0}(V_t^*\alpha)$.

Proof: By definition of differential, $\mathrm{Lie}_{v_0} f = \langle df, v_0\rangle$, hence $\mathrm{Lie}_{v_0} f = \frac{d}{dt}|_{t=0} V_t^*(f)$. The operator Lie_{v_0} commutes with de Rham differential, because $\mathrm{Lie}_v = i_v d + d i_v$. The map $\frac{d}{dt}V_t$ commutes with de Rham differential, because it is a derivative of a pullback. Now **Lemma (*) is applied to show that** $\mathrm{Lie}_{v_0}\alpha = \frac{d}{dt}|_{t=0}(V_t^*\alpha)$. ■

Homotopy operators

DEFINITION: A complex is a sequence of vector spaces and homomorphisms $... \xrightarrow{d} C_{i-1} \xrightarrow{d} C_i \xrightarrow{d} C_{i+1} \xrightarrow{d} ...$
satisfying $d^2 = 0$. **Homomorphism** $(C_*, d) \longrightarrow (C'_*, d)$ of complexes is a sequence of homomorphism $C_i \longrightarrow C'_i$ commuting with the differentials.

DEFINITION: An element $c \in C_i$ is called **closed** if $c \in \ker d$ and **exact** if $c \in \operatorname{im} d$. **Cohomology** of a complex is a quotient $\frac{\ker d}{\operatorname{im} d}$.

REMARK: A homomorphism of complexes induces a natural homomorphism of cohomology groups.

DEFINITION: Let (C_*, d), (C'_*, d) be a complex. **Homotopy** is a sequence of maps $h: C_* \longrightarrow C'_{*-1}$. Two homomorphisms $f, g: (C_*, d) \longrightarrow (C'_*, d)$ are called **homotopy equivalent** if $f - g = \{h, d\}$ for some homotopy operator h.

CLAIM: Let $f, f': (C_*, d) \longrightarrow (C'_*, d)$ be homotopy equivalent maps of complexes. **Then f and f' induce the same maps on cohomology.**

Proof. Step 1: Let $g := f - f'$. It would suffice to prove that g induces 0 on cohomology.

Lie derivative and homotopy

CLAIM: Let $f, f' : (C_*, d) \longrightarrow (C'_*, d)$ be homotopy equivalent maps of complexes. **Then f and f' induce the same maps on cohomology.**

Proof. Step 1: Let $g := f - f'$. It would suffice to prove that g induces 0 on cohomology.

Step 2: Let $c \in C_i$ be a closed element. **Then** $g(c) = dh(c) + hd(c) = dh(c)$ **exact.** ■

DEFINITION: Let d be de Rham differential. A form in $\ker d$ is called **closed**, a form in $\operatorname{im} d$ is called **exact**. Since $d^2 = 0$, any exact form is closed. **The group of i-th de Rham cohomology of M**, denoted $H^i(M)$, is a quotient of a space of closed i-forms by the exact: $H^*(M) = \frac{\ker d}{\operatorname{im} d}$.

REMARK: Let v be a vector field, and $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$ be the corresponding Lie derivative. Then Lie_v **commutes with the de Rham differential, and acts trivially on the de Rham cohomology.**

Proof: $\operatorname{Lie}_v = i_v d + d i_v$ maps closed forms to exact. ■

Poincaré lemma

DEFINITION: An open subset $U \subset \mathbb{R}^n$ is called **starlike** if for any $x \in U$ the interval $[0, x]$ belongs to U.

THEOREM: (Poicaré lemma) Let $U \subset \mathbb{R}^n$ be a starlike subset. **Then $H^i(U) = 0$ for $i > 0$.**

REMARK: The proof would follow if we construct a vector field $\vec{r}$ such that $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^*(M)$: $\operatorname{Lie}_{\vec{r}} R = \operatorname{Id}$. Indeed, for any closed form α we would have $\alpha = \operatorname{Lie}_{\vec{r}} R\alpha = di_{\vec{r}}R\alpha + i_{\vec{r}}Rd\alpha = di_{\vec{r}}R\alpha$, hence any closed form is exact.

Then Poincaré lemma is implied by the following statement.

PROPOSITION: Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, ..., t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. **Then $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^i(U)$ for $i > 0$.**

Radial vector field on starlike sets

PROPOSITION: Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, ..., t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. **Then $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^i(U)$ for $i > 0$.**

Proof. Step 1: Let t be the coordinate function on a real line, $f(t) \in C^\infty\mathbb{R}$ a smooth function, and $v := t\frac{d}{dt}$ a vector field. Define $R(f)(t) := \int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda$. Then this integral converges whenever $f(0) = 0$, and satisfies $\operatorname{Lie}_v R(f) = f$. Indeed,

$$\int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda = \int_0^t \frac{f(\lambda t)}{t\lambda} d(t\lambda) = \int_0^t \frac{f(z)}{z} dz,$$

hence $\operatorname{Lie}_v R(f) = t\frac{f(t)}{t} = f(t)$.

Step 2: Consider a function $f \in C^\infty\mathbb{R}^n$ satisfying $f(0) = 0$, and $x = (x_1, ..., x_n) \in \mathbb{R}^n$. **Then**

$$R(f)(x) := \int_0^1 \frac{f(\lambda x)}{\lambda} d\lambda$$

converges, and satisfies $\operatorname{Lie}_{\vec{r}} R(f) = f$.

Radial vector field on starlike sets (2)

Step 3: Consider a differential form $\alpha \in \Lambda^i$, and let $h_\lambda x \longrightarrow \lambda x$ be the homothety with coefficient $\lambda \in [0,1]$. Define

$$R(\alpha) := \int_0^1 \lambda^{-1} h_\lambda^*(\alpha) d\lambda.$$

Since $h_\lambda^*(\alpha) = 0$ for $\lambda = 0$, this integral converges. **It remains to prove that $\operatorname{Lie}_{\vec{r}} R = \operatorname{Id}$.**

Step 4: Let α be a coordinate monomial, $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$. Clearly, $\operatorname{Lie}_{\vec{r}}(T^{-1}\alpha) = 0$, where $T = t_{i_1} t_{i_2} ... t_{i_k}$. **Since $h_\lambda^*(f\alpha) = h_\lambda^*(Tf)T^{-1}\alpha$, we have $R(f\alpha) = R(Tf)T^{-1}\alpha$ for any function $f \in C^\infty M$.** This gives

$$\operatorname{Lie}_{\vec{r}} R(f\alpha) = \operatorname{Lie}_{\vec{r}} R(Tf)T^{-1}\alpha = TfT^{-1}\alpha = f\alpha.$$

■