Riemann surfaces: final exam

Rules: Every student receives from me a list of 6 exercises (chosen randomly), and has to solve 3 of them by January 17. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solution, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes). Maximal score is 3 out of 6 exercises, but every exercise you solve brings you closer. Feel free to google the solutions if you are able. Problems with "k points" are worth k usual exercises.

1 Almost complex structures and Hodge decomposition

Exercise 1.1. Let ρ be a 2-form on a real vector space equipped with a complex structure operator $I \in \operatorname{End}(V)$, $I^2 = -\operatorname{Id}$. Assume that $\rho(x, Iy) = \rho(Ix, y)$ for all $x, y \in V$. Prove that ρ is a real part of a $(2,0)$-form.

Exercise 1.2. Let ω be a non-degenerate 2-form on a real manifold M. Prove that there exists an almost complex structure I on M and a Hermitian form g such that ω is its Hermitian form, $\omega = g(\cdot, I\cdot)$.

Definition 1.1. Let M be an almost complex manifold, $A: \Lambda^* M \longrightarrow \Lambda^* M$ an endomorphism of the space of differential forms. **Hodge components** of A are operators $A^{p,q}$ such that $A = \sum_{p,q} A^{p,q}$ and $A^{p,q}(\Lambda^{i,j}(M)) \subset \Lambda^{i+p,j+q}(M)$.

Exercise 1.3. Prove that de Rham differential on an almost complex manifold has no more than 4 Hodge components: $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$.

Exercise 1.4. Let $\tau: M \longrightarrow M$ be an involutive diffeomorphism of an almost complex manifold (M, I), $\tau^* I = -I$, and M^τ its fixed point set. Prove that M^τ is a smooth submanifold, and $\dim_{\mathbb{R}} M^\tau = \frac{1}{2} \dim_{\mathbb{R}} M$.

2 Holomorphic functions

Exercise 2.1. Let f a real-valued smooth function on a complex manifold which satisfies $dId(f) = 0$. Prove that f is a real part of a holomorphic function.

Exercise 2.2. Let f be a holomorphic function on disk $\Delta \subset \mathbb{C}$ which is continuoulsly extended to the boundary $\partial\Delta$. Suppose that $f\big|_{\partial\Delta}$ takes values in $\mathbb{R}$. Prove that f is constant, or find a counterexample.

Exercise 2.3. Let (M, I) be an almost complex 2-manifold, admitting a function f with $dIdf = 0$. Assume that $df \neq 0$ on a dense subset of M. Prove that the almost complex structure on M is integrable,

Exercise 2.4. Let (M, I) be a Hermitian Riemann surface, ω its Hermitian form, and f a function which satisfies $dIdf = 0$, and B an open ball of radius r with respect to the Riemannian distance and center in x. Prove that

$$f(x) = \frac{\int_B f\omega}{\int_B \omega}.$$

3 Differential forms on complex manifolds

Exercise 3.1. Let M be a complex manifold, and θ a non-zero exact form such that $d(I\theta) = 0$. Prove that M admits a non-zero holomorphic function.

Exercise 3.2. Let η be a holomorphic $n-1$-form on a compact complex manifold of complex dimension n. Prove that $d\eta = 0$.

Exercise 3.3. Let f be a real function on a Riemannian surface M such that the top-form $dIdf$ is proportional to the volumme form with non-negative coefficient.[1] Prove that that f cannot have a strict maximum anywhere on M.

Exercise 3.4 (2 points). Let f, g be real functions on a Riemannian surface M with boundary ∂M. Assume that $f \geqslant g$ on the boundary of M and the corresponding 2-forms $dIdf$ and $dIdg$ satisfy $dIdf \leqslant dIdg$ everywhere (we can compare these forms because they are top-forms, that is, proportional to the volume form). Prove that $f \geqslant g$ on M.

4 Homogeneous spaces

Exercise 4.1. Let (M, I) be a smooth almost complex manifold equipped with a transitive action of a group G. Assume that I is G-invariant (such a manifold is called **homogeneous**). Assume, moreover, that for some $x \in M$ there exists an element $\tau_x \in G$ fixing x. Consider the induced action of τ_x on T_xM.

a. Suppose that $\tau_x = \lambda \operatorname{Id}$, where $\lambda \in \mathbb{R}$. Prove that for all $\lambda \neq 1$, the almost complex structure I is integrable.

b. Construct examples of such (M, I), G and τ_x for each $\lambda \in \mathbb{R}$.

Exercise 4.2. Construct a G-invariant almost complex structure on a homogeneous manifold G/H which is not integrable.

5 Automorphism groups

Exercise 5.1. Let M be a Kobayashi hyperbolic, compact Riemann surface. Prove that the group $\operatorname{Aut}(M)$ of holomorphic automorphisms of M is finite.

Exercise 5.2. Let M be a Riemann surface of constant curvature, with universal covering isometric to the Poincare disk Δ, giving $M = \Delta/\Gamma$, where $\Gamma = \pi_1(M)$ is a group acting properly and freely on Δ. Let $M = \Delta/\Gamma$ be a Riemann surface of constant curvature such that its automorphism group is infinite. Prove that $\pi_1(\Delta) = \mathbb{Z}$.

Exercise 5.3. Let M be a Riemann surface of constant curvature, with universal covering isometric to the Poincare disk Δ, giving $M = \Delta/\Gamma$, where $\Gamma = \pi_1(M)$. Consider the action of Γ on the boundary $\partial\Delta$. A point $x \in \partial\Delta$ is called **a cusp point** if it is fixed by some element $\gamma \in \Gamma$ of infinite order. Assume that M is of finite volume.

a. (2 points) Prove that M is compact if and only if there are no cusp points.

b. Prove that Γ acts on the set of cusp points with finite number of orbits.

6 Poincare metric

Exercise 6.1. Let Δ be a disk in $\mathbb{C}$ and M a complex manifold which is Kobayashi hyperbolic. Prove that any holomorphic map $\Psi : \Delta \longrightarrow M$ can be continuously extended to the boundary of Δ.

Exercise 6.2 (2 points). Let $\Delta^* = \Delta\backslash 0$ be a disk in $\mathbb{C}$ without 0 and M a complex manifold which is Kobayashi hyperbolic. Prove that any holomorphic map $\Psi : \Delta^* \longrightarrow M$ can be continuously extended to 0.

[1] Such forms are called **positive**.