Riemann surfaces

lecture 1

Misha Verbitsky

Université Libre de Bruxelles

September 23, 2015

Complex structure on vector spaces

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

We extend the action of I on the tensor spaces $V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$ by multiplicativity: $I(v_1 \otimes ... \otimes w_1 \otimes ... \otimes w_n) = I(v_1) \otimes ... \otimes I(w_1) \otimes ... \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues α_i of I are $\pm\sqrt{-1}$.** Indeed, $\alpha_i^2 = -1$.

2. **V admits an I-invariant, positive definite scalar product ("metric") g.** Take any metric g_0, and let $g := g_0 + I(g_0)$.

3. **I is orthogonal for such g.**
Indeed, $g(Ix, Iy) = g_0(x, y) + g_0(Ix, Iy) = g(x, y)$.

4. **I diagonalizable over $\mathbb{C}$.** Indeed, any orthogonal matrix is diagonalizable.

Hermitian structures

5. **There are as many $\sqrt{-1}$-eigenvalues as there are $-\sqrt{-1}$-eigenvalues.**

Denote by ν the **real structure operator,** $\nu(\sum \lambda_i w_i) = \sum \overline{\lambda}_i w_i$, where $w_i \in V$ is a basis. Then $\nu(I(z)) = I(\nu(z))$, that is, I **is real**. For any $\sqrt{-1}$-eigenvector w, one has $I(\nu(w)) = \nu(I(w)) = \nu(\sqrt{-1}\, w) = -\sqrt{-1}\, w$, hence ν **exchanges $\sqrt{-1}$-eigenvectors and $-\sqrt{-1}$-eigenvectors.**

DEFINITION: An I-invariant positive definite scalar product on (V, I) is called **an Hermitian metric**, and (V, I, g) – an Hermitian space.

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x, y) = g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -\omega(y, x)$.

DEFINITION: A skew-symmetric form $\omega(x, y)$ is called **an Hermitian form on (V, I)**.

REMARK: In the triple I, g, ω, each element can recovered from the other two.

The Grassmann algebra

DEFINITION: Let V be a vector space. Denote by $\Lambda^i V$ the space of antisymmetric polylinear i-forms on V^*, and let $\Lambda^* V := \bigoplus \Lambda^i V$. Denote by $T^{\otimes i} V$ the algebra of **all** polylinear i-forms on V^* ("tensor algebra"), and let $\operatorname{Alt}: T^{\otimes i} V \longrightarrow \Lambda^i V$ be **the antisymmetrization**,

$$\operatorname{Alt}(\eta)(x_1, ..., x_i) := \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (-1)^{\tilde{\sigma}} \eta(x_{\sigma_1}, ..., x_{\sigma_i})$$

where Σ_i is the group of permutations, and $\tilde{\sigma} = 1$ for odd permutations, and 0 for even. Consider the multiplicative operation ("wedge-product") on $\Lambda^* V$, denoted by $\eta \wedge \nu := \operatorname{Alt}(\eta \otimes \nu)$. The space $\Lambda^* V$ with this operation is called **the Grassmann algebra**.

REMARK: It is an algebra of anti-commutative polynomials.

Properties of Grassmann algebra:

1. $\dim \Lambda^i V := \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.

2. $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$.

The Hodge decomposition in linear algebra

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda^n_{\mathbb{R}} V \otimes_{\mathbb{R}} C$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMARK: The operator I induces $U(1)$-action on V by the formula $\rho(t)(v) = \cos t \cdot v + \sin t \cdot I(v)$. We extend this action on the tensor spaces by muptiplicativity.

$U(1)$-representations and the weight decomposition

REMARK: Any complex representation W of $U(1)$ is written as a sum of 1-dimensional representations $W_i(p)$, with $U(1)$ acting on each $W_i(p)$ as $\rho(t)(v) = e^{\sqrt{-1}\,pt}(v)$. The 1-dimensional representations are called **weight p representations of $U(1)$.**

DEFINITION: A **weight decomposition** of a $U(1)$-representation W is a decomposition $W = \oplus W^p$, where each $W^p = \oplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p.

REMARK: The Hodge decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a weight decomposition, with $\Lambda^{p,q} V$ being a weight $p-q$-component of $\Lambda^n V_{\mathbb{C}}$.

REMARK: $V^{p,p}$ is the space of $U(1)$-invariant vectors in $\Lambda^{2p} V$.

Further on, **TM is the tangent bundle on a manifold, and $\Lambda^i M$ the space of differential i-forms.** It is a Grassmann algebra on TM.

Vector fields

DEFINITION: Let X be the vector field on a manifold M, and f a function. Denote by $\operatorname{Lie}_X f$ **the derivatiive** of f along X.

DEFINITION: **A derivation** on a commutative ring is a map $R \xrightarrow{d} R$ satisfying **the Leibniz identity** $d(xy) = d(x)y + xd(y)$.

THEOREM: Each derivation of the ring $C^\infty M$ of smooth functions on M is given by a vector field X; **this correspondence is bijective.**

REMARK: This can be used as a definition of a vector field.

EXERCISE: Prove that **a commutator of two derivations is again a derivation.**

REMARK: Vector fields are the same as derivations of $C^\infty M$. This allows us to define **the commutator of two vector fields** as the commutator of the corresponding derivations.

DEFINITION: Denote by TM the bundle of vector fields, and by $\Lambda^1 M$ or T^* the dual bundle, called **the bundle of 1-forms**. For any $f \in C^\infty M$, the operation $X \longrightarrow \operatorname{Lie}_X f$ is linear as a function of X, hence it defines a section of T^*M. We denote this section df, and call it **the differential** of f.

De Rham algebra

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T^*_x M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T^*_x M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: **There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ satisfying the following properties**

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

EXERCISE: **Prove it**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Sheaves

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and presheaves: examples

Examples of sheaves:

* Space of continuous functions

* Space of smooth functions, any differentiability class

* Space of real analytic functions

Examples of presheaves which are not sheaves:

* Space of constant functions **(why?)**

* Space of bounded functions **(why?)**

Ringed spaces

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

EXAMPLE: Let M be a manifold of class C^i and let $C^i(U)$ be the space of functions of this class. **Then C^i is a sheaf of functions, and (M, C^i) is a ringed space.**

REMARK: Let $f : X \longrightarrow Y$ be a smooth map of smooth manifolds. Since a pullback $f^*\mu$ of a smooth function $\mu \in C^\infty(M)$ is smooth, **a smooth map of smooth manifolds defines a morphism of ringed spaces.**

Complex manifolds

DEFINITION: A holomorphic function on $\mathbb{C}^n$ is a function $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: A complex manifold M is a ringed space which is locally isomorphic to an open ball in $\mathbb{C}^n$ with a sheaf of holomorphic functions.

REMARK: In other words, M **is covered with open balls embedded to** $\mathbb{C}^n$ and transition functions (being coordinate functions for one ball considered in other coordinate system) **are holomorphic.**