Riemann surfaces

lecture 3

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De Rham algebra (reminder)

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

THEOREM: There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ satisfying the following properties

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Stokes' theorem: Let η be $n-1$-form on n-manifold M with a boundary ∂M. **Then $\int_M d\eta = \int_{\partial M} \eta$.**

The Hodge decomposition in linear algebra (reminder)

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

We extend the action of I on the tensor spaces $V\otimes V\otimes ... \otimes V\otimes V^*\otimes V^*\otimes ... \otimes V^*$ by multiplicativity: $I(v_1\otimes ... \otimes w_1\otimes ... \otimes w_n) = I(v_1)\otimes ... \otimes I(w_1)\otimes ... \otimes I(w_n)$.

DEFINITION: Let (V,I) be a space equipped with a complex structure. **The Hodge decomposition** $V\otimes_{\mathbb{R}}\mathbb{C} := V^{1,0}\oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb{C}} := V\otimes_{\mathbb{R}}\mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda^n_{\mathbb{R}} V\otimes_{\mathbb{R}} C$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0}\otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0}\otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q}V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n}\Lambda^{p,q}V$ is called **the Hodge decomposition of the Grassmann algebra**.

Holomorphic functions (reminder)

DEFINITION: Let $I: TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

DEFINITION: A function $f: M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally. Example: S^6 with a certain canonical (G_2-invariant) complex structure.

Holomorphic functions on $\mathbb{C}^n$ (reminder)

THEOREM: Let $f: M \longrightarrow \mathbb{C}$ be a differentiable function on an open subset $M \subset \mathbb{C}^n$, with the natural almost complex structure. **Then the following are equivalent.**

(1) f **is holomorphic**.

(2) The differential $df: TM \longrightarrow \mathbb{C}$, considered as a form on the vector space $T_xM = T_x\mathbb{C}^n = \mathbb{C}^n$ **is $\mathbb{C}$-linear.**

(3) For any complex affine line $L \in \mathbb{C}^n$, the restriction $f|_L = \mathbb{C}$ is **holomorphic (complex analytic) as a function of one complex variable.**

(4) f is expressed as a sum of Taylor series around any point $(z_1, ..., z_n) \in M$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the complex numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

Sheaves (reminder)

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Ringed spaces (reminder)

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

EXAMPLE: Let M be a manifold of class C^i and let $C^i(U)$ be the space of functions of this class. **Then C^i is a sheaf of functions, and (M, C^i) is a ringed space.**

Complex manifolds (reminder)

DEFINITION: A holomorphic function on $\mathbb{C}^n$ is a function $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: A complex manifold M is a ringed space which is locally isomorphic to an open ball in $\mathbb{C}^n$ with a sheaf of holomorphic functions.

REMARK: In other words, M **is covered with open balls embedded to** $\mathbb{C}^n$ and transition functions (being coordinate functions for one ball considered in other coordinate system) **are holomorphic.**

Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. ■

Frobenius form (reminder)

CLAIM: Let $B \subset TM$ be a sub-bundle of a tangent bundle of a smooth manifold. Given vector fiels $X, Y \in B$, consider their commutator $[X, Y]$, and lets $\Psi(X, Y) \in TM/B$ be the projection of $[X, Y]$ to TM/B. **Then $\Psi(X, Y)$ is $C^\infty(M)$-linear in X, Y:**

$$\Psi(fX, Y) = \Psi(X, fY) = f\Psi(X, Y).$$

Proof: Leibnitz identity gives $[X, fY] = f[X, Y] + X(f)Y$, and the second term belongs to B, hence does not influence the projection to TM/B. ■

DEFINITION: This form is called **the Frobenius form** of the sub-bundle $B \subset TM$. This bundle is called **involutive**, or **integrable**, or **holonomic** if $\Psi = 0$.

EXERCISE: Give an example of a non-integrable sub-bundle.

Formal integrability (reminder)

DEFINITION: An almost complex structure I on (M,I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^{2,0}M \otimes TM$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg)
A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) later in this lecture.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.

Real analytic manifolds

DEFINITION: Real analytic function on an open set $U \subset \mathbb{R}^n$ is a function which admits Taylor expansion near each point $x \in U$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the real numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

REMARK: Clearly, **real analytic functions constitute a sheaf.**

DEFINITION: A **real analytic manifold** is a ringed space which is locally isomorphic to an open ball $B \subset \mathbb{R}^n$ with the sheaf of of real analytic functions.

Involutions

DEFINITION: An involution is a map $\iota: M \longrightarrow M$ such that $\iota^2 = \operatorname{Id}_M$.

EXERCISE: Prove that **any linear involution on a real vector space V is diagonalizable,** with eigenvalues ± 1.

Theorem 1: Let M be a smooth manifold, and $\iota: M \longrightarrow M$ an involutiin. **Then the fixed point set N of ι is a smooth submanifold.**

Proof. Step 1: Inverse function theorem. Let $m \in M$ be a point on a smooth k-dimensional manifold and $f_1, ..., f_k$ functions on M such that their differentials $df_1, ..., df_k$ are linearly independent in m. Then $f_1, ..., f_k$ **define a coordinate system in a neighbourhood of a, giving a diffeomorphism of this neighbourhood to an open ball.**

Step 2: Assume that $d\iota$ has k eigenvalues 1 on $T_m M$, and $n-k$ eigenvalues -1. Choose a coordinate system $x_1, ..., x_n$ on M around a point $m \in N$ such that $dx_1|_m, ..., dx_k|_m$ are ι-invariant and $dx_{k+1}|_m, ..., dx_n|_m$ are ι-anti-invariant. Let $y_1 = x_1 + \iota^* x_1$, $y_2 = x_2 + \iota^* x_2$, ... $y_k = x_k + \iota^* x_k$, and $y_{k+1} = x_{k+1} - \iota^* x_{k+1}$, $y_{k+2} = x_{k+2} - \iota^* x_{k+2}$, ... $y_n = x_n - \iota^* x_n$. Since $dx_i|_m = xy_i|_m$, these differentials are linearly independent in m. By Step 1, **functions y_i define an ι-invariant coordinate system on an open neighbourhood of m, with N given by equations $y_{k+1} = y_{k+2} = ... = y_n = 0$.** ■

Real structures

DEFINITION: An involution is a map $\iota : M \longrightarrow M$ such that $\iota^2 = \operatorname{Id}_M$. **A real structure** on a complex vector space $V = \mathbb{C}^n$ is an $\mathbb{R}$-linear involution $\iota : V \longrightarrow V$ such that $\iota(\lambda x) = \overline{\lambda}\iota(x)$ for any $\lambda \in \mathbb{C}$.

DEFINITION: A map $\Psi : M \longrightarrow M$ on an almost complex manifold (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

EXERCISE: Let ι be a smooth map from a complex manifold M to itself. Prove that **ι is antiholomorphic if and only if $\iota^*(f)$ is antiholomorphic for any holomorphic function f on $U \subset M$.**

DEFINITION: A real structure on a complex manifold M is an antiholomorphic involution $\tau : M \longrightarrow M$.

EXAMPLE: Complex conjugation defines a real structure on $\mathbb{C}^n$.

Real analytic manifolds and real structures

PROPOSITION: Let $M_{\mathbb R} \subset M_{\mathbb C}$ be a fixed point set of an antiholomorphic involution ι, U_i a complex analytic atlas, and $\Psi_{ij}: U_{ij} \longrightarrow U_{ij}$ the gluing functions. **Then, for appropriate choice of coordinate systems all Ψ_{ij} are real analytic on $M_{\mathbb R}$, and define a real analytic atlas on the manifold $M_{\mathbb R}$.**

Proof. Step 1: Let $z_1, ..., z_n$ be a holomorphic coordinate system on $M_{\mathbb C}$ in a neighbourhood of $m \in M_{\mathbb R}$ such that $\iota(dz_i) = d\overline{z}_i$ in $T^*_m M$. Such a coordinate system can be chosen by taking linear functions with prescribed differentials in m. **Replacing z_i by $x_i := z_i + \iota^*(\overline{z}_i)$, we obtain another coordinate system x_i on M** (compare with Theorem 1).

Step 2: This new coordinate system satisfies $\iota^* x_i = \overline{x}_i$, hence $M_{\mathbb R}$ in these coordinates is giving by equation $\operatorname{im} x_1 = \operatorname{im} x_2 = ... = \operatorname{im} x_n = 0$. **All gluing functions from such coordinate system to another one of this type satisfy $\Psi_{ij}(\overline{z}_i) = \overline{\Psi_{ij}(\overline{z}_i)}$, hence they are real on $M_{\mathbb R}$.** ■

Real analytic manifolds and real structures (2)

PROPOSITION: Any real analytic manifold can be obtained from this construction.

Proof. Step 1: Let $\{U_i\}$ be a locally finite atlas of a real analytic manifold M, and $\Psi_{ij} : U_{ij} \longrightarrow U_{ij}$ the gluing map. We realize U_i as an open ball with compact closure in Re$(\mathbb{C}^n) = \mathbb{R}^n$. By local finiteness, there are only finitely many such Ψ_{ij} for any given U_i. Denote by B_ε an open ball of radius ε in the n-dimensional real space im$(\mathbb{C}^n)$.

Step 2: Let $\varepsilon > 0$ be a sufficiently small real number such that all Ψ_{ij} can be extended to gluing functions $\tilde\Psi_{ij}$ on the open sets $\tilde U_i := U_i \times B_\varepsilon \subset \mathbb{C}^n$. **Then $(\tilde U_i, \Psi_{ij})$ is an atlas for a complex manifold $M_{\mathbb{C}}$.** Since all Ψ_{ij} are real, they are preserved by natural involution acting on B_ε as -1 and on U_i as identity. This involution defines a real structure on $M_{\mathbb{C}}$. Clearly, M is the set of its fixed points. ■

Complexification

DEFINITION: Let $M_{\mathbb R}$ be a real analytic manifold, and $M_{\mathbb C}$ a complex analytic manifold equipped with an antiholomorphic involution, such that $M_{\mathbb R}$ is the set of its fixed points. Then $M_{\mathbb C}$ is called **complexification** of $M_{\mathbb R}$.

DEFINITION: A tensor on a real analytic manifold is called **real analytic** if it is expressed locally by a sum of coordinate monomials with real analytic coefficients.

CLAIM: Let $M_{\mathbb R}$ be a real analytic manifold, $M_{\mathbb C}$ its complexification, and Φ a tensor on $M_{\mathbb R}$. **Then Φ is real analytic if and only if Φ can be extended to a holomorpic tensor $\Phi_{\mathbb C}$ in some neighbourhood of $M_{\mathbb R}$ inside $M_{\mathbb C}$.**

Proof: The "if" part is clear, because every complex analytic tensor on $M_{\mathbb C}$ is by definition real analytic on $M_{\mathbb R}$.

Conversely, suppose that Φ is expressed by a sum of coordinate monomials with real analytic coefficients f_i. Let $\{U_i\}$ be a cover of M, and $\tilde U_i := U_i \times B_\varepsilon$ the corresponding cover of a neighbourhood of $M_{\mathbb R}$ in $M_{\mathbb C}$ constructed above. Chosing ε sufficiently small, we can assume that the Taylor series giving coefficients of Φ converges on each $\tilde U_i$. **We define $\Phi_{\mathbb C}$ as the sum of these series.** ■

Extension of tensors to a complexification

Lemma 1: Let X be an open ball in $\mathbb{C}^n$ equipped with the standard anticomplex involution, $X_{\mathbb{R}} = X \cap \mathbb{R}^n$ its fixed point set, and α a holomorphic tensor on X vanishing in $X_{\mathbb{R}}$. **Then $\alpha = 0$.**

Proof: Any holomorphic function which vanishes on $\mathbb{R}^n$ has all its derivatives is equal zero. Therefore its Taylor series vanish. Such a function vanishes on $\mathbb{C}^n$ by analytic continuation principle. This argument can be applied to all coefficients of α. ■

DEFINITION: An almost complex structure I on a real analytic manifold is **real analytic** if I is a real anaytic tensor.

COROLLARY: Let (M, I) be a real analytic almost complex manifold, $M_{\mathbb{C}}$ its complexification, and $I_{\mathbb{C}} : TM_{\mathbb{C}} \longrightarrow TM_{\mathbb{C}}$ the holomorphic extension of I to $M_{\mathbb{C}}$. **Then $I_{\mathbb{C}}^2 = -\mathrm{Id}$.**

Proof: The tensor $I_{\mathbb{C}}^2 + \mathrm{Id}$ is holomorphic and vanishes on $M_{\mathbb{R}}$, hence the previous lemma can be applied. ■

Underlying real analytic manifold

REMARK: A complex analytic map $\Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is real analytic as a map $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$. Indeed, the coefficients of Φ are real and imaginary parts of holomorphic functions, and real and imaginary parts of holomorphic functions can be expressed as Taylor series of the real variables.

DEFINITION: Let M be a complex manifold. The **underlying real analytic manifold** is the same manifold, with the same gluing functions, considered as real analytic maps.

DEFINITION: Let M be a complex manifold. The **complex conjugate manifold** is the same manifold with almost complex structure $-I$ and anti-holomorphic functions on M holomorphic on $\overline{M}$.

CLAIM: Let M be an integrable almost complex manifold. Denote by $M_\mathbb{R}$ its underlying real analytic manifold. **Then a complexification of $M_\mathbb{R}$ can be given as $M_\mathbb{C} := M \times \overline{M}$,** with the anticomplex involution $\tau(x, y) = (y, x)$.

Proof: Clearly, the fixed point set of τ is the diagonal, identified with $M_\mathbb{R} = M$ as usual. Both holomorphic and antiholomorphic functions on $M_\mathbb{R}$ are obtained as restrictions of holomorphic functions from $M_\mathbb{C}$, hence the sheaf of real analytic functions on $M_\mathbb{R}$ is a real part of the sheaf $\mathcal{O}_{M_\mathbb{C}}$ of holomorphic functions on $M_\mathbb{C}$. ■