Riemann surfaces

lecture 4

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De Rham algebra (reminder)

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

THEOREM: There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} \dots$ satisfying the following properties

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Stokes' theorem: Let η be $n-1$-form on n-manifold M with a boundary ∂M. **Then $\int_M d\eta = \int_{\partial M} \eta$.**

The Hodge decomposition in linear algebra (reminder)

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

We extend the action of I on the tensor spaces $V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$ by multiplicativity: $I(v_1 \otimes ... \otimes w_1 \otimes ... \otimes w_n) = I(v_1) \otimes ... \otimes I(w_1) \otimes ... \otimes I(w_n)$.

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda^n_{\mathbb{R}} V \otimes_{\mathbb{R}} C$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

Holomorphic functions (reminder)

DEFINITION: Let $I: TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

DEFINITION: A function $f: M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally. Example: S^6 with a certain canonical (G_2-invariant) complex structure.

Holomorphic functions on $\mathbb{C}^n$ (reminder)

THEOREM: Let $f: M \longrightarrow \mathbb{C}$ be a differentiable function on an open subset $M \subset \mathbb{C}^n$, with the natural almost complex structure. **Then the following are equivalent.**

(1) f **is holomorphic**.

(2) The differential $df: TM \longrightarrow \mathbb{C}$, considered as a form on the vector space $T_x M = T_x\mathbb{C}^n = \mathbb{C}^n$ **is $\mathbb{C}$-linear.**

(3) For any complex affine line $L \in \mathbb{C}^n$, the restriction $f|_L = \mathbb{C}$ is **holomorphic (complex analytic) as a function of one complex variable.**

(4) f is expressed as a sum of Taylor series around any point $(z_1, ..., z_n) \in M$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the complex numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

Sheaves (reminder)

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Ringed spaces (reminder)

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \stackrel{\Psi}{\longrightarrow} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \stackrel{\Psi}{\longrightarrow} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

EXAMPLE: Let M be a manifold of class C^i and let $C^i(U)$ be the space of functions of this class. **Then C^i is a sheaf of functions, and (M, C^i) is a ringed space.**

Complex manifolds (reminder)

DEFINITION: A holomorphic function on $\mathbb{C}^n$ is a function $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: A complex manifold M is a ringed space which is locally isomorphic to an open ball in $\mathbb{C}^n$ with a sheaf of holomorphic functions.

REMARK: In other words, M **is covered with open balls embedded to** $\mathbb{C}^n$ and transition functions (being coordinate functions for one ball considered in other coordinate system) **are holomorphic.**

Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. ■

Frobenius form (reminder)

CLAIM: Let $B \subset TM$ be a sub-bundle of a tangent bundle of a smooth manifold. Given vector fiels $X, Y \in B$, consider their commutator $[X, Y]$, and lets $\Psi(X, Y) \in TM/B$ be the projection of $[X, Y]$ to TM/B. **Then $\Psi(X, Y)$ is $C^\infty(M)$-linear in X, Y:**

$$\Psi(fX, Y) = \Psi(X, fY) = f\Psi(X, Y).$$

Proof: Leibnitz identity gives $[X, fY] = f[X, Y] + X(f)Y$, and the second term belongs to B, hence does not influence the projection to TM/B. ■

DEFINITION: This form is called **the Frobenius form** of the sub-bundle $B \subset TM$. This bundle is called **involutive**, or **integrable**, or **holonomic** if $\Psi = 0$.

EXERCISE: Give an example of a non-integrable sub-bundle.

Formal integrability (reminder)

DEFINITION: An almost complex structure I on (M,I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^{2,0}M \otimes TM$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg)
A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) later in this lecture.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.

Real analytic manifolds (reminder)

DEFINITION: Real analytic function on an open set $U \subset \mathbb{R}^n$ is a function which admits Taylor expansion near each point $x \in U$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the real numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

REMARK: Clearly, **real analytic functions constitute a sheaf.**

DEFINITION: A **real analytic manifold** is a ringed space which is locally isomorphic to an open ball $B \subset \mathbb{R}^n$ with the sheaf of of real analytic functions.

Real structures (reminder)

DEFINITION: An involution is a map $\iota: M \longrightarrow M$ such that $\iota^2 = \operatorname{Id}_M$.

EXERCISE: Prove that **any linear involution on a real vector space V is diagonalizable,** with eigenvalues ± 1.

Theorem 1: Let M be a smooth manifold, and $\iota: M \longrightarrow M$ an involutiin. **Then the fixed point set N of ι is a smooth submanifold.**

DEFINITION: A real structure on a complex vector space $V = \mathbb{C}^n$ is an $\mathbb{R}$-linear involution $\iota: V \longrightarrow V$ such that $\iota(\lambda x) = \overline{\lambda} x$ for any $\lambda \in \mathbb{C}$.

DEFINITION: A map $\Psi: M \longrightarrow M$ on an almost complex manifold (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

DEFINITION: A real structure on a complex manifold M is an antiholomorphic involution $\tau: M \longrightarrow M$.

EXAMPLE: Complex conjugation defines a real structure on $\mathbb{C}^n$.

Real analytic manifolds and real structures (reminder)

PROPOSITION: Let $M_{\mathbb{R}} \subset M_{\mathbb{C}}$ be a fixed point set of an antiholomorphic involution τ, U_i a complex analytic atlas which is preserved by τ, and $\Psi_{ij}: U_{ij} \longrightarrow U_{ij}$ the gluing functions. **Then all Ψ_{ij} are real analytic on $M_{\mathbb{R}}$, and define a real analytic atlas on the manifold $M_{\mathbb{R}}$.**

PROPOSITION: **Any real analytic manifold can be obtained from this construction.**

Proof. Step 1: Let $\{U_i\}$ be a locally finite atlas of a real analytic manifold M, and $\Psi_{ij}: U_{ij} \longrightarrow U_{ij}$ the gluing map. We realize U_i as an open ball with compact closure in Re$(\mathbb{C}^n) = \mathbb{R}^n$. By local finiteness, there are only finitely many such Ψ_{ij} for any given U_i. Denote by B_ε an open ball of radius ε in the n-dimensional real space im$(\mathbb{C}^n)$.

Step 2: Let $\varepsilon > 0$ be a sufficiently small real number such that all Ψ_{ij} can be extended to gluing functions $\tilde{\Psi}_{ij}$ on the open sets $\tilde{U}_i := U_i \times B_\varepsilon \subset \mathbb{C}^n$. **Then $(\tilde{U}_i, \Psi_{ij})$ is an atlas for a complex manifold $M_{\mathbb{C}}$.** Since all Ψ_{ij} are real, they are preserved by natural involution acting on B_ε as -1 and on U_i as identity. This involution defines a real structure on $M_{\mathbb{C}}$. Clearly, M is the set of its fixed points. ■

Complexification (reminder)

DEFINITION: Let $M_{\mathbb{R}}$ be a real analytic manifold, and $M_{\mathbb{C}}$ a complex analytic manifold equipped with an antiholomorphic involution, such that $M_{\mathbb{R}}$ is the set of its fixed points. Then $M_{\mathbb{C}}$ is called **complexification** of $M_{\mathbb{R}}$.

DEFINITION: A tensor on a real analytic manifold is called **real analytic** if it is expressed locally by a sum of coordinate monomials with real analytic coefficients.

DEFINITION: An almost complex structure I on a real analytic manifold is **real analytic** if I is a real anaytic tensor.

CLAIM: Let (M, I) be a real analytic almost complex manifold, $M_{\mathbb{C}}$ its complexification, and $I_{\mathbb{C}}: TM_{\mathbb{C}} \longrightarrow TM_{\mathbb{C}}$ the holomorphic extension of I to $M_{\mathbb{C}}$. **Then $I_{\mathbb{C}}^2 = -\operatorname{Id}$.**

Proof: **The tensor $I_{\mathbb{C}}^2 + \operatorname{Id}$ is holomorphic and vanishes on $M_{\mathbb{R}}$,** hence the analytic continuation principle can be applied. ■

Underlying real analytic manifold (reminder)

REMARK: **A complex analytic map $\Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is real analytic as a map $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$.** Indeed, the coefficients of Φ are real and imaginary parts of holomorphic functions, and real and imaginary parts of holomorphic functions can be expressed as Taylor series of the real variables.

DEFINITION: Let M be a complex manifold. The **underlying real analytic manifold** $M_{\mathbb{R}}$ is the same topological manifold, with the same gluing functions, considered as real analytic maps.

REMARK: The real analytic functions on $M_{\mathbb{R}}$ are obtained as converging series of holomorphic and antiholomorphic variables.

Complexification of the underlying real analytic manifold (reminder)

DEFINITION: Let M be a complex manifold. The **complex conjugate manifold** is the same manifold with almost complex structure $-I$.

CLAIM: Let M be an integrable almost complex manifold. Denote by $M_{\mathbb R}$ its underlying real analytic manifold. **Then a complexification of $M_{\mathbb R}$ can be given as $M_{\mathbb C} := M \times \overline{M}$,** with the anticomplex involution $\tau(x, y) = -(y, x)$.

Proof: Clearly, the fixed point set of τ is the diagonal, identified with $M_{\mathbb R} = M$ as usual. Both holomorphic and antiholomorphic functions on $M_{\mathbb R}$ are obtained as restrictions of holomorphic functions from $M_{\mathbb C}$, hence the sheaf of real analytic functions on $M_{\mathbb R}$ is a real part of the sheaf $\mathcal{O}_{M_{\mathbb C}}$ of holomorphic functions on $M_{\mathbb C}$. ■

Holomorphic and antiholomorphic foliations

DEFINITION: Let $B \subset TM$ be a sub-bundle. The **foliation associated with** B is a family of submanifolds $X_t \subset M$, called **the leaves of the foliation** such that B is the bundle of vectors tangent to the leaves X_t.

REMARK: The famous "Frobenius theorem" says that B **is involutive if and only if it is tangent to a foliation.**

REMARK: Let (M, I) be a real analytic almost complex manifold, and $M_{\mathbb C}$ its complexification. Replacing $M_{\mathbb C}$ by a smaller neighbourhood of M, we may assume that the tensor I is extended to an endomorphism $I: TM_{\mathbb C} \longrightarrow TM_{\mathbb C}$, $I^2 = -\operatorname{Id}$. **Since $TM_{\mathbb C}$ is a complex vector bundle, I acts there with the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, giving a decomposition $TM_{\mathbb C} = T^{1,0}M_{\mathbb C} \oplus T^{0,1}M_{\mathbb C}$**

DEFINITION: Holomorphic foliation is a foliation tangent to $T^{1,0}M_{\mathbb C}$, **antiholomorphic foliation** is a foliation tangent to $T^{0,1}M_{\mathbb C}$.

Antiholomorphic foliation on $M_{\mathbb C} = M \times \overline{M}$.

CLAIM: Let (M,I) be a integrable almost complex manifold, $M_{\mathbb C} = M \times \overline{M}$ its complexification, and $\pi, \overline{\pi}$ projections of $M_{\mathbb C}$ to M and $\overline{M}$. **Then the fibers of $\overline{\pi}$ is a holomorphic foliation, and the fibers of π is a holomorphic foliation.**

Proof: Let $TM_{\mathbb C} = T' \oplus T''$ be a decomposition of $TM_{\mathbb C}$ onto part tangent to fibers of $\overline{\pi}$ and tangent to fibers of π. **On $M_{\mathbb R}$ the decomposition $TM_{\mathbb C} = T' \oplus T''$ coincides with the decomposition $TM \otimes \mathbb C = T^{1,0}M \oplus T^{0,1}M$.** By Lemma 1 the same is true everywhere on $M_{\mathbb C}$. ■

COROLLARY: Let (M,I) be a integrable almost complex manifold. **Then I is a real analytic almost complex structure.**

Proof: It was extended to $M_{\mathbb C}$ in the previous claim. ■

Corollary 1: Let (M,I) be a real analytic almost complex manifold. Then holomorphic functions on $M_{\mathbb C}$ which are constant on the leaves of antiholomoirphic foliation **restrict to holomorphic functions on $(M,I) \subset M_{\mathbb C}$.**

Proof: Such functions are constant in the $(0,1)$-direction on $TM \otimes \mathbb C$. ■

Integrability of real analytic almost complex structure

THEOREM: ("linearization of a vector field") Let $v \in TM$ be a nowhere vanishing vector field on M. **Then there exists a family of 1-dimensional submanifolds passing through each point of M such that v is tangent to these submanifolds at each point of M.**

THEOREM: Let (M, I) be a real analytic almost complex manifold, $\dim_{\mathbb{R}} M = 2$. **Then M is integrable.**

Proof. Step 1: Consider the complexification $M_{\mathbb{C}}$ of M, and let $TM_{\mathbb{C}} = T^{1,0}M_{\mathbb{C}} \oplus T^{0,1}M_{\mathbb{C}}$ be the decomposition defined above. By "linearization of a vector field" theorem, there exists a foliation tangent to $T^{0,1}M_{\mathbb{C}}$ and one tangent to $T^{1,0}M_{\mathbb{C}}$. Since the leaves of these foliations are transversal, **locally $M_{\mathbb{C}}$ is a product of M' and M'' which are identified with the space of leaves of $T^{0,1}M_{\mathbb{C}}$ and $T^{1,0}M_{\mathbb{C}}$.**

Step 2: Locally, functions on M' can be lifted to $M' \times M'' = M_{\mathbb{C}}$, giving functions which are constant on the leaves of the foliation tangent to $T^{0,1}M_{\mathbb{C}}$. By Corollary 1, such functions are holomorphic on (M, I). Choosing a function with linearly independent differentials in $x \in M$, it would give a **holomorphic coordinate system in a neigbourhood of (M, I),** and the transition functions between such coordinate systems are by construction holomorphic.

■

Riemannian manifolds

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x.y \in M$, and any path $\gamma: [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Hermitian structures

DEFINITION: A Riemannia metric h on an almost complex manifold is called **Hermitian** if $h(x,y) = h(Ix, Iy)$.

REMARK: Given any Riemannian metric g on an almost complex manifold, **a Hermitian metric h can be obtained as $h = g + I(g)$, where $I(g)(x,y) = g(I(x), I(y))$.**

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x,y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x,y) = g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -\omega(y,x)$.

DEFINITION: A skew-symmetric form $\omega(x,y)$ is called **an Hermitian form on (V, I)**.

REMARK: In the triple I, g, ω, each element can recovered from the other two.

Conformal structure

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

REMARK: The last statement is clear from the definition, and true in any dimension.

To prove that two Hermitian metrics are equivalent, we need to consider the standard $U(1)$-action on a complex vector space (see the next slide).

Standard $U(1)$-action

DEFINITION: Let (V, I) be a real vector space equipped with a complex structure, $U(1)$ the group of unit complex numbers, $U(1) = e^{\sqrt{-1}\,\pi t}$, $t \in \mathbb{R}$. Define the action of $U(1)$ on V as follows: $\rho(t) = e^{tI}$. This is called **the standard $U(1)$-action on a complex vector space**. To prove that this formula defines an action if $U(1) = \mathbb{R}/2\pi\mathbb{Z}$, it suffices to show that $e^{2\pi I} = 1$, which is clear from the eigenvalue decomposition of I.

CLAIM: Let (V, I, h) be a Hermitian vector space, and $\rho : U(1) \longrightarrow GL(V)$ the standard $U(1)$-action. **Then h is $U(1)$-invariant.**

Proof: It suffices to show that $\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = 0$. However, $\frac{d}{dt}e^{tI}(x)\Big|_{t=t_0} = I(e^{t_0 I}(x))$, hence

$$\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = h(I(\rho(t)x), \rho(t)x) + h(\rho(t)x, I(\rho(t)x)) = 2\omega(x, x) = 0.$$

■

Hermitian metrics in $\dim_{\mathbb{R}} = 2$.

COROLLARY: Let h, h' be Hermitian metrics on a space (V, I) of real dimension 2. **Then h and h' are proportional.**

Proof: h and h' are constant on any $U(1)$-orbit. Multiplying h' by a constant, we may assume that $h = h'$ on a $U(1)$-orbit $U(1)x$. Then $h = h'$ everywhere, because **for each non-zero vector $v \in V$, $tv \in U(1)x$ for some $t \in \mathbb{R}$, giving $h(v, v) = t^{-2} h(tv, tv) = t^{-2} h'(tv, tv) = h'(v, v)$.** ■

DEFINITION: Given two Hermitian forms h, h' on (V, I), with $\dim_{\mathbb{R}} V = 2$, we denote by $\frac{h'}{h}$ a constant t such that $h' = th$.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**.

Proof: $h' = \frac{h'}{h} h$. ■

EXERCISE: Prove that **Riemannian structure on M is uniquely defined by its conformal class and its Riemannian volume form**.

Conformal structures and almost complex structures

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1})$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure. A map from one Riemannian surface to another is holomorphic if and only if it preserves the conformal structure.