Riemann surfaces

lecture 5: conformal structures

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Riemannian manifolds (reminder)

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x.y \in M$, and any path $\gamma: [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Hermitian structures (reminder)

DEFINITION: A Riemannia metric h on an almost complex manifold is called **Hermitian** if $h(x, y) = h(Ix, Iy)$.

REMARK: Given any Riemannian metric g on an almost complex manifold, **a Hermitian metric h can be obtained as $h = g + I(g)$, where $I(g)(x, y) = g(I(x), I(y))$.**

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x, y) = g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -\omega(y, x)$.

DEFINITION: A skew-symmetric form $\omega(x, y)$ is called **an Hermitian form on (V, I)**.

REMARK: In the triple I, g, ω, each element can recovered from the other two.

Conformal structure (reminder)

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

REMARK: The last statement is clear from the definition, and true in any dimension.

To prove that two Hermitian metrics are equivalent, we need to consider the standard $U(1)$-action on a complex vector space (see the next slide).

Standard $U(1)$-action

DEFINITION: Let (V, I) be a real vector space equipped with a complex structure, $U(1)$ the group of unit complex numbers, $U(1) = e^{\sqrt{-1}\,\pi t}$, $t \in \mathbb{R}$. Define the action of $U(1)$ on V as follows: $\rho(t) = e^{tI}$. This is called **the standard $U(1)$-action on a complex vector space**. To prove that this formula defines an action if $U(1) = \mathbb{R}/2\pi\mathbb{Z}$, it suffices to show that $e^{2\pi I} = 1$, which is clear from the eigenvalue decomposition of I.

CLAIM: Let (V, I, h) be a Hermitian vector space, and $\rho: U(1) \longrightarrow GL(V)$ the standard $U(1)$-action. **Then h is $U(1)$-invariant.**

Proof: It suffices to show that $\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = 0$. However, $\frac{d}{dt}e^{tI}(x)\Big|_{t=t_0} = I(e^{t_0 I}(x))$, hence

$$\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = h(I(\rho(t)x), \rho(t)x) + h(\rho(t)x, I(\rho(t)x)) = 2\omega(x, x) = 0.$$

■

Hermitian metrics in $\dim_{\mathbb{R}} = 2$.

COROLLARY: Let h, h' be Hermitian metrics on a space (V, I) of real dimension 2. **Then h and h' are proportional.**

Proof: h and h' are constant on any $U(1)$-orbit. Multiplying h' by a constant, we may assume that $h = h'$ on a $U(1)$-orbit $U(1)x$. Then $h = h'$ everywhere, because **for each non-zero vector $v \in V$, $tv \in U(1)x$ for some $t \in \mathbb{R}$, giving $h(v, v) = t^{-2}h(tv, tv) = t^{-2}h'(tv, tv) = h'(v, v)$.** ∎

DEFINITION: Given two Hermitian forms h, h' on (V, I), with $\dim_{\mathbb{R}} V = 2$, we denote by $\frac{h'}{h}$ a constant t such that $h' = th$.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**.

Proof: $h' = \frac{h'}{h} h$. ∎

EXERCISE: Prove that **Riemannian structure on M is uniquely defined by its conformal class and its Riemannian volume form**.

Conformal structures and almost complex structures

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1})$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure. A map from one Riemann surface to another is holomorphic if and only if it preserves the conformal structure.

Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\operatorname{St}_x(G) = \{g \in G \ \mid \ g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \operatorname{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\operatorname{St}_x(G)^g = \operatorname{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A tensor Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on $\mathrm{St}_x(G)$. For any $y \in M$ obtained as $y = g(x)$, consider the tensor Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

THEOREM: Homogeneous tensors on $M = G/H$ are in bijective correspondence with isotropy invariant tensors on T_xM, for any $x \in M$. ■

Space forms

DEFINITION: Simply connected space form is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

Riemannian metric on space forms

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form,** the standard Euclidean.

Proof: Let g, g' be two G-invariant symmetric 2-forms. Since S^{n-1} is an orbit of G, we have $g(x,x) = g(y,y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x,x) = g'(x,x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: **From now on, all space forms are assumed to be homogeneous Riemannian manifolds**.

Some low-dimensional Lie group isomorphisms

DEFINITION: Lie algebra of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields. **Adjoint representation** of G is the standard action of G on Lie(G). For a Lie group $G = GL(n)$, $SL(n)$, etc., $PGL(n)$, $PSL(n)$, etc. denote the image of G in $GL(\text{Lie}(G))$ with respect to the adjoint action.

REMARK: This is the same as a quotient G/Z by the centre of G.

DEFINITION: Define $SO(1,2)$ as the group of orthogonal matrices on a 3-dimensional space equipped with a scalar product of signature (1,2), and $U(1,1)$ as the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$ and $SO(1,2)$ are isomorphic.

Proof: Isomorphism $PU(1,1) = SO(1,2)$ will be established later in this lecture. To see $PSL(2,\mathbb{R}) \cong SO(1,2)$, consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a, b \longrightarrow \text{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO(1,2)$.** Both groups are 3-dimensional, hence it is an isomorphism. ■

Poincaré-Koebe uniformization theorem

DEFINITION: A **Riemannian manifold of constant curvature** is a Riemannian manifold which is locally isomorphic to a space form.

THEOREM: (Poincaré-Koebe uniformization theorem) Let M be a Riemann surface. **Then M admits a unique complete metric of constant curvature in the same conformal class.**

COROLLARY: Any Riemann surface is a quotient of a space form X by a discrete group of isometries $\Gamma \subset \mathrm{Iso}(X)$.

COROLLARY: Any simply connected Riemann surface is conformally equivalent to a space form.

REMARK: We shall prove some cases of the uniformization theorem in later lectures.

Today's subject: classify conformal automorphisms of all space forms.

Laurent power series

THEOREM: (Laurent theorem)
Let f be a holomorphic function on an annulus (that is, a ring)

$$R = \{z \mid \alpha < |z| < \beta\}.$$

Then f can be expressed as a Laurent power series $f(z) = \sum_{i\in\mathbb{Z}} z^i a_i$ converging in R.

Proof: Same as Cauchy formula. ■

REMARK: This theorem remains valid if $\alpha = 0$ and $\beta = \infty$.

REMARK: A function $\varphi: \mathbb{C}^* \longrightarrow \mathbb{C}$ uniquely determines its Laurent power series. Indeed, residue of $z^k\varphi$ in 0 is $\sqrt{-1}\, 2\pi a_{-k-1}$.

REMARK: Let $\varphi: \mathbb{C}^* \longrightarrow \mathbb{C}$ be a holomorphic function, and $\varphi = \sum_{i\in\mathbb{Z}} z^i a_i$ its Laurent power series. **Then $\psi(z) := \varphi(z^{-1})$ has Laurent polynomial $\psi = \sum_{i\in\mathbb{Z}} z^{-i} a_i$.**

Affine coordinates on $\mathbb{C}P^1$

DEFINITION: We identify $\mathbb{C}P^1$ with the set of pairs $x : y$ defined up to equivalence $x : y \sim \lambda x : \lambda y$, for each $\lambda \in \mathbb{C}^*$. **Affine coordinates** are $1 : z$ for $x \neq 0$, $z = y/x$ and $z : 1$ for $y \neq 0$, $z = x/y$. The corresponding gluing functions are given by the map $z \longrightarrow z^{-1}$.

DEFINITION: **Meromorphic function** is a quotient f/g, where f, g are holomorphic and $g \neq 0$.

REMARK: A holomorphic map $\mathbb{C} \longrightarrow \mathbb{C}P^1$ is the same as a pair of maps $f : g$ up to equivalence $f : g \sim fh : gh$. **In other words, holomorphic maps $\mathbb{C} \longrightarrow \mathbb{C}P^1$ are identified with meromorphic functions on $\mathbb{C}$.**

REMARK: In homogeneous coordinates, an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$ acts as $x : y \longrightarrow ax + by : cx + dy$. Therefore, in affine coordinates it acts as $z \longrightarrow \frac{az+b}{cz+d}$.

Möbius transforms

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.

The following theorem will be proven later in this lecture.

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group of Möbius transforms is an isomorphism.

Claim 1: Let $\varphi : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic automorphism, $\varphi_0 : \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction to the chart $z : 1$, and $\varphi_\infty : \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction $1 : z$. We consider φ_0, φ_∞ as meromorphic functions on $\mathbb{C}$. **Then $\varphi_\infty = \varphi_0(z^{-1})^{-1}$.**

Möbius transforms and $PGL(2,\mathbb{C})$

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group $\operatorname{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

Proof. Step 1: Let $\varphi \in \operatorname{Aut}(\mathbb{C}P^1)$. Since $PSL(2,\mathbb{C})$ acts transitively on pairs of points $x \neq y$ in $\mathbb{C}P^1$, by composing φ with an appropriate element in $PGL(2,\mathbb{C})$ we can assume that $\varphi(0) = 0$ and $\varphi(\infty = \infty$. This means that we may consider the restrictions φ_0 and φ_∞ of φ to the affine charts as a holomorphic functions on these charts, $\varphi_0, \varphi_\infty : \mathbb{C} \longrightarrow \mathbb{C}$.

Step 2: Let $\varphi_0 = \sum_{i>0} a_i z^i$, $a_1 \neq 0$. Claim 1 gives

$$\varphi_\infty(z) = \varphi_0(z^{-1})^{-1} = a_1 z(1 + \sum_{i\geqslant 2} \frac{a_i}{a_1} z^{-i})^{-1}.$$

Unless $a_i = 0$ for all $i \geqslant 2$, this Laurent series has singularities in 0 and cannot be holomorphic. **Therefore φ_0 is a linear function**, and it belongs to $PGL(2,\mathbb{C})$. ■

Lemma 1: Let φ be a Möbius transform fixing $\infty \in \mathbb{C}P^1$. **Then $\varphi(z) = az+b$ for some $a, b \in \mathbb{C}$ and all $z = z : 1 \in \mathbb{C}P^1$.**

Proof: Let $A \in PGL(2,\mathbb{C})$ be a map acting on $\mathbb{C} = \mathbb{C}P^1 \backslash \infty$ as parallel transport mapping $\varphi(0)$ to 0. Then $\varphi \circ A$ is a Moebius transform which fixes ∞ and 0. As shown in Step 2 above, it is a linear function. ■